August 2018

Internal and External Harmonic Functions in Flat-Ring Coordinates

Lijuan Bi
University of Wisconsin-Milwaukee

INTERNAL AND EXTERNAL HARMONIC FUNCTIONS IN FLAT-RING COORDINATES

by

Lijuan Bi

A Dissertation Submitted in
Partial Fulfillment of the
Requirements for the Degree of

Doctor of Philosophy
in
Mathematics

at

The University if Wisconsin-Milwaukee
August 2018

ABSTRACT

INTERNAL AND EXTERNAL HARMONIC FUNCTIONS IN FLAT-RING COORDINATES

by

Lijuan Bi

The University of Wisconsin-Milwaukee, 2018
Under the Supervision of Professor Hans Volkmer

The goal of this dissertation is to derive expansions for a fundamental solution of Laplace's equation in flat-ring coordinates in three-dimensional Euclidean space. These expansions are in terms of harmonic functions in the interior and the exterior of two different types of regions, "flat rings" and "peanuts" according to their shapes. We solve Laplace's equation in the interior and the exterior of these regions using the method of separation of variables. The internal and external "flat-ring" and "peanut" harmonic functions are expressed in terms of Lamé functions.

TABLE OF CONTENTS

ACKNOWLEDGMENTS

I would like to thank my advisor, Professor Hans Volkmer, for his help and patience. I appreciate the efforts of Professor Rebecca Bourn, who is always willing to help me and cheer me up. I would also like to express my gratitude to all my dissertation committee members, Professors Peter Hinow, Gabriella Pinter, Jeb Willenbring, and Dexuan Xie, for their help. Last but not least, I would like to thank my parents, friends, and other professors for their encouragement.

Chapter 1

Introduction

It has been known for a long time that Laplace's equation in cartesian coordinates on $\mathbb{R}^3$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1.0.1}$$

can be solved in a quadric or a cyclidic coordinate system on $\mathbb{R}^3$ using separation of variables methods. A cyclidic coordinate system in $\mathbb{R}^3$ has three coordinates, each coordinate is obtained by a transformation from cartesian coordinates in $\mathbb{R}^3$, and the coordinate surfaces are cyclides. A cyclide in general is a fourth-order surface, which is a natural extension of a quadric surface. A quadric surface in general is a second-order surface.

Bôcher listed quadric and cyclidic coordinate systems where Laplace's equation can be solved using separation of variables in his book "Ueber die Reihenentwicklungen der Potentialtheorie" [9] published in 1894. We can also find these quadric and cyclidic coordinates systems in Miller's Book ([4], Table 14 on page 164, Table 17 on page 210). The quadric coordinates are cartesian, cylindrical, parabolic cylindrical, elliptic cylindrical, spherical, prolate spheroidal, oblate spheroidal, parabolic, paraboloidal, conical, and ellipsoidal coordinates. The cyclidic coordinates are flat-ring cyclide, flat-disk cyclide, bi-cyclide, cap-cyclide, and 3-cyclide coordinates.

Using separation of variables methods, a solution of (1.0.1) can be written in the form

$$u(x, y, z) = R(\alpha, \beta, \lambda)v_1(\alpha)v_2(\beta)v_3(\lambda)$$

where α, β, λ are quadric or cyclidic coordinates, $R(\alpha, \beta, \lambda)$ is a fixed and known function, and $v_1(\alpha), v_2(\beta), v_3(\lambda)$ are solutions of a second order linear homogeneous ordinary differential equation, respectively. We have $R(\alpha, \beta, \lambda) = 1$ if α, β, λ are quadric coordinates, and $R(\alpha, \beta, \lambda) \neq 1$, if α, β, λ are cyclidic coordinates. For example, we solve the (1.0.1) in a well-known quadric coordinate system, the spherical coordinate system,

$$x = r\,\cos\phi\,\sin\theta,\;\; y = r\,\sin\phi\,\sin\theta,\;\; z = r\,\cos\theta, \tag{1.0.2}$$

$$r > 0,\ -\pi < \phi < \pi,\ 0 < \theta < \pi.$$

The equation (1.0.1) has solutions of the form

$$u(x, y, z) = u_1(r)u_2(\theta)u_3(\phi).$$

The u_1, u_2, u_3 have to satisfy the following equations, respectively:

$$r^2 u_1'' + 2ru_1' - n(n+1)u_1 = 0, \tag{1.0.3}$$

$$u_2'' + \cot\theta u_2' + \left[n(n+1) - \frac{m^2}{\sin^2\theta}\right] u_2 = 0, \tag{1.0.4}$$

$$u_3'' + m^2 u_3 = 0, \tag{1.0.5}$$

where m, n are separation parameters. A general solution of (1.0.3) is

$$u_1(r) = c_1 r^n + c_2 r^{-n-1},$$

where c_1 and c_2 are constants. A general solution of (1.0.4) is

$$u_2(\theta) = c_1 P_n^m(\cos\theta) + Q_n^m(\cos\theta),$$

where P_n^m and Q_n^m are the associated Legendre functions of the first kind and the second kind, respectively. We consider the simplest cyclidic coordinate system, the toroidal coordinate system,

$$x = \frac{\sinh\sigma\cos\phi}{\cosh\sigma - \cos\psi},\ y = \frac{\sinh\sigma\sin\phi}{\cosh\sigma - \cos\psi},\ z = \frac{\sin\psi}{\cosh\sigma - \cos\psi}.$$

The coordinate surfaces $\sigma = \textit{constant}$ are tori, i.e.

$$(1 + x^2 + y^2 + z^2)^2 = 4(x^2 + y^2)\coth^2\sigma.$$

The coordinate surfaces $\psi = \textit{constant}$ are spherical bowls, i.e.

$$(z - \cot\psi)^2 + x^2 + y^2 = \frac{1}{\sin^2\psi}.$$

The coordinate surfaces $\phi = \textit{constant}$ are planes, i.e.

$$x\sin\phi = y\cos\phi.$$

We assume

$$u(x, y, z) = \sqrt{\cosh\sigma - \cos\psi}\ u_1(\sigma)u_2(\phi)u_3(\psi).$$

We find u_1, u_2, u_3 satisfies each of the following equations, respectively:

$$\frac{1}{\sinh\sigma}(\sinh\sigma u_1')' - \left(n^2 - \frac{1}{4} + \frac{m^2}{\sinh^2\sigma}\right)u_1 = 0, \tag{1.0.6}$$

$$u_2'' + n^2 u_2 = 0, \tag{1.0.7}$$

$$u_3'' + m^2 u_3 = 0, \tag{1.0.8}$$

where m, n are separation parameters. A general solution of (1.0.6) is

$$u_1(\sigma) = c_1 P^m_{n-\frac{1}{2}}(\cosh\sigma) + c_2 Q^m_{n-\frac{1}{2}}(\cosh\sigma),$$

where c_1 and c_2 are constants, and P^m_n and Q^m_n are the associated Legendre functions of the first kind and the second kind, respectively. Special functions we obtain from solving Laplace's equation in a quadric coordinate system via separation of variables are functions like Bessel functions, Legendre functions, Lamé polynomials, etc, which are some of the most classical special functions. For a cyclidic coordinate, special functions are functions like Legendre functions, Lamé periodic functions, Lamé-Wangerin functions, solutions of an ordinary equation with five elementary singularities (see [6]), etc.

The Helmholtz equation on $\mathbb{R}^3$,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0,$$

where k^2 is a real number, can be solved via separation of variables in the same quadric coordinate systems that Laplace's equation can be solved using separation of variables. However, the Helmholtz equation cannot be solved in the cyclidic coordinate systems listed above using separation of variables.

It is also long known that a fundamental solution of Laplace's equation in $\mathbb{R}^3$, i.e.

$$\frac{1}{||\mathbf{r}-\mathbf{r}'||} = \left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{-\frac{1}{2}},$$

where $\mathbf{r} = (x, y, z)$, $\mathbf{r}' = (x', y', z')$, and $\mathbf{r} \neq \mathbf{r}'$, can be expressed using solutions from separation of variables (see [10], [11], [4], and [12]). For example, we consider spherical coordinates. The internal spherical harmonics are

$$G^m_n(x, y, z) = r^n P^m_n(\cos\theta)e^{im\phi}, \ -n \leq m \leq n,$$

where P^m_n is an associated Legendre function of the first kind (Ferrer's function). G^m_n is a harmonic function in $\mathbb{R}^3$. The external spherical harmonics are

$$H^m_n(x, y, z) = r^{-n-1} P^m_n(\cos\theta)e^{im\phi}, \ -n \leq m \leq n.$$

H_n^m is a harmonic function in $\mathbb{R}^3$ except the origin. An expansion of a fundamental solution is

$$\frac{1}{||\mathbf{r}-\mathbf{r}'||} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{(n-m)!}{(n+m)!} \left(G_n^m(\mathbf{r}) \overline{H_n^m(\mathbf{r}')} \right), \tag{1.0.9}$$

where $||\mathbf{r}|| < ||\mathbf{r}'||$. As another example, we can consider toroidal coordinates. The internal toroidal harmonics are

$$G_n^m(x,y,z) = \sqrt{\cosh\sigma - \cos\psi}\, Q^m_{n-\frac{1}{2}}(\cosh\sigma) e^{in\psi} e^{im\phi},$$

where $m, n \in \mathbb{Z}$ and $Q^m_{n-\frac{1}{2}}$ is the associated Legendre function of the second kind. G_n^m is harmonic in $\mathbb{R}^3$ except for the z-axis. The external toroidal harmonics are

$$H_n^m(x,y,z) = \sqrt{\cosh\sigma - \cos\psi}\, P^m_{n-\frac{1}{2}}(\cosh\sigma) e^{in\psi} e^{im\phi},$$

where $m, n \in \mathbb{Z}$ and $P^m_{n-\frac{1}{2}}$ is the associated Legendre function of the first kind. H_n^m is harmonic in $\mathbb{R}^3$ except for the unit circle $z = 0$, $x^2 + y^2 = 1$.

In this dissertation we consider flat-ring coordinates which is listed as number 15 in Miller's list [9, page 210] (see [4]). The coordinates ρ, μ, ϕ in algebraic form compose an orthogonal coordinate system in $\mathbb{R}^3$ with rotational symmetry. They vary according to $1 < \rho < a < \mu < \infty$, where a is a given number. For our convenience, we use flat-ring coordinates α, β, ϕ in transcendental form, which is a transformation of the algebraic form. The surface $\alpha = constant$ looks similar to a flat-ring. The surface $\beta = constant$ looks similar to a peanut. We derive expansions analogous to (1.0.9) for flat-ring coordinate system in terms of flat-ring harmonics and peanut harmonics, respectively. As far as we know, the expansions are given for the first time.

The outline of this dissertation is as follows. In chapter 2 we discuss flat-ring coordinates. In chapter 3 we solve Laplace's equation in flat-ring coordinates using the method of separation of variables. The solutions are in terms of Lamé functions. In chapter 4 we collect some facts about Lamé functions that will be used in the later chapters. In chapter 5 we derive internal and external flat-ring harmonics, solve the Dirichlet problem and provide expansions for a fundamental solution of Laplace's equation in terms of internal and external flat-ring harmonics. In chapter 6 we derive internal and external peanut harmonics, solve the Dirichlet problem and provide an expansion for a fundamental solution of Laplace's equation in terms of internal and external peanut harmonics.

This introduction is based on Cohl's manuscript (see [2]), the introduction of [6], and the introduction of [7].

Chapter 2

Flat-Ring Coordinates

2.1 Flat-Ring Coordinates in Algebraic Form

Flat-ring coordinates ρ, μ, ϕ form an orthogonal coordinate system in $\mathbb{R}^3$ with rotational symmetry. According to ([4], system 15, page 210), it is given by

$$x = \frac{\cos\phi}{r}, \qquad y = \frac{\sin\phi}{r}, \qquad z = \frac{1}{r}\left(\frac{(\mu-a)(a-\rho)}{a(a-1)}\right)^{\frac{1}{2}}, \tag{2.1.1}$$

where

$$r = \left(\frac{\mu\rho}{a}\right)^{\frac{1}{2}} + \left(\frac{(\mu-1)(\rho-1)}{a-1}\right)^{\frac{1}{2}}. \tag{2.1.2}$$

The number $a > 1$ is given. The coordinates ρ, μ, ϕ vary according to

$$1 < \rho < a < \mu < \infty.$$

The square roots appearing (2.1.1) and (2.1.2) are understood to be positive.

It is easier to visualize in two dimensions. So we set $\phi = 0$ to remove rotational symmetry and look at the two-dimensional coordinate system as a slice. Now

$$x = \frac{1}{r}, \qquad z = \frac{1}{r}\left(\frac{((\mu-a)(a-\rho)}{a(a-1)}\right)^{\frac{1}{2}} \tag{2.1.3}$$

where

$$r = \left(\frac{\mu\rho}{a}\right)^{\frac{1}{2}} + \left(\frac{(\mu-1)(\rho-1)}{a-1}\right)^{\frac{1}{2}}. \tag{2.1.4}$$

We will show the mapping $(x, z) = f(\rho, \mu)$ according to (2.1.3) and (2.1.4) is bijective.

Proposition 1. *If $A = (1,a) \times (a,\infty)$, $Q_0 = \{(x,z) : x > 0, z > 0, x^2+z^2 < 1\}$, and $(x,z) = f(\rho,\mu)$ according to (2.1.3) and (2.1.4), we obtain that f maps A to Q_0 and*

$$\frac{(x^2+z^2+1)^2}{s} - \frac{(x^2+z^2-1)^2}{s-1} - \frac{4z^2}{s-a} = 0, \tag{2.1.5}$$

where $s = \mu$ or ρ.

Proof. First we show that f maps A to Q_0. Let $(\rho,\mu) \in A$ and $(x,z) = f(\rho,\mu)$ according to (2.1.3) and (2.1.4). Since the square roots in (2.1.3) and (2.1.4) are understood to be positive, we have $x > 0$ and $z > 0$. We wish to show $x^2+z^2 < 1$. According to (2.1.4), we have

$$r = \left(\frac{\mu\rho}{a}\right)^{\frac{1}{2}} + \left(\frac{(\mu-1)(\rho-1)}{a-1}\right)^{\frac{1}{2}}.$$

We find

$$\frac{1}{r} = \frac{1}{\left(\frac{\mu\rho}{a}\right)^{\frac{1}{2}} + \left(\frac{(\mu-1)(\rho-1)}{a-1}\right)^{\frac{1}{2}}} = \frac{\left(\frac{\mu\rho}{a}\right)^{\frac{1}{2}} - \left(\frac{(\mu-1)(\rho-1)}{a-1}\right)^{\frac{1}{2}}}{\left(\frac{\mu a-\mu\rho+\rho a-a}{a(a-1)}\right)}. \tag{2.1.6}$$

We also calculate

$$x^2+z^2-1 = \frac{1}{r^2} + \frac{1}{r^2}\left(\frac{(\mu-a)(a-\rho)}{a(a-1)}\right) - 1 \tag{2.1.7}$$

$$= \frac{1}{r^2}\left(\frac{\mu a-\mu\rho+\rho a-a}{a(a-1)}\right) - 1. \tag{2.1.8}$$

Using the results from (2.1.6) and (2.1.8), we obtain

$$\begin{aligned}\frac{x^2+z^2-1}{\frac{1}{r}} &= \frac{\frac{1}{r^2}\left(\frac{\mu a-\mu\rho+\rho a-a}{a(a-1)}\right) - 1}{\frac{1}{r}} \\ &= \frac{1}{r}\left(\frac{\mu a-\mu\rho+\rho a-a}{a(a-1)}\right) - r \\ &= -2\left(\frac{(\mu-1)(\rho-1)}{a-1}\right)^{\frac{1}{2}} < 0.\end{aligned}$$

It is clear that

$$x^2+z^2-1 < 0, \tag{2.1.9}$$

because

$$\frac{1}{r} > 0.$$

Hence f maps A to Q_0. Second, we wish to show equation (2.1.5) is true.
We consider

$$r^2\left[\frac{(x^2+z^2+1)^2}{\mu}-\frac{(x^2+z^2-1)^2}{\mu-1}-\frac{4z^2}{\mu-a}\right]$$
$$=\frac{\left(\frac{x^2+z^2+1}{\frac{1}{r}}\right)^2}{\mu}-\frac{\left(\frac{x^2+z^2-1}{\frac{1}{r}}\right)^2}{\mu-1}-4\ \frac{a-\rho}{a(a-1)}$$
$$=\frac{\left(2\ \left(\frac{\mu\rho}{a}\right)^{\frac{1}{2}}\right)^2}{\mu}-\frac{\left(2\ \left(\frac{(\mu-1)(\rho-1)}{a-1}\right)^{\frac{1}{2}}\right)^2}{\mu-1}-4\ \frac{a-\rho}{a(a-1)}$$
$$=4\ \left[\frac{\rho}{a}-\frac{\rho-1}{a-1}-\frac{a-\rho}{a(a-1)}\right]=0.$$

So it is clear that
$$\frac{(x^2+z^2+1)^2}{\mu}-\frac{(x^2+z^2-1)^2}{\mu-1}-\frac{4z^2}{\mu-a}=0.$$

With a similar calculation, we have
$$\frac{(x^2+z^2+1)^2}{\rho}-\frac{(x^2+z^2-1)^2}{\rho-1}-\frac{4z^2}{\rho-a}=0.$$

□

Proposition 2. *If $A=(1,a)\times(a,\infty)$, $Q_0=\{(x,z):x,z>0,x^2+z^2<1\}$, and $(x,z)=f(\rho,\mu)$ according to (2.1.3) and (2.1.4), then $f:A\to Q_0$ is bijective.*

Proof. Let $(x,z)\in Q_0$. Assume

$$F(s)=s(s-1)(s-a)\left[\frac{(x^2+z^2+1)^2}{s}-\frac{(x^2+z^2-1)^2}{s-1}-\frac{4z^2}{s-a}\right], \tag{2.1.10}$$

The coefficient of s^2 is

$$(x^2+z^2+1)^2-(x^2+z^2-1)^2-4z^2=4x^2>0. \tag{2.1.11}$$

Moreover,

$$\lim_{s\to 0}F(s)=a(x^2+z^2+1)^2>0,$$
$$\lim_{s\to 1}F(s)=(a-1)(x^2+z^2-1)^2>0,$$
$$\lim_{s\to a}F(s)=-a(a-1)4z^2<0.$$

If $F(s) = 0$, we can find two real solutions, say ρ and μ, such that $1 < \rho < a$ and $\mu > a$ according to (2.1.11) and the inequalities above. Then we obtain

$$F(s) = 4x^2(s-\rho)(s-\mu), \tag{2.1.12}$$

where s is a real number. According to (2.1.12), we get

$$\begin{aligned}
&\lim_{s\to 0} F(s) = 4x^2\rho\mu, \\
&\lim_{s\to 1} F(s) = 4x^2(1-\rho)(1-\mu), \\
&\lim_{s\to a} F(s) = 4x^2(a-\rho)(a-\mu).
\end{aligned}$$

We match the above corresponding limits respectively and we get

$$a(x^2+z^2+1)^2 = 4x^2\rho\mu, \tag{2.1.13}$$

$$(a-1)(x^2+z^2-1)^2 = 4x^2(1-\rho)(1-\mu), \tag{2.1.14}$$

$$-a(a-1)4z^2 = 4x^2(a-\rho)(a-\mu). \tag{2.1.15}$$

From equation (2.1.13), we have

$$\frac{\rho\mu}{a} = \frac{(x^2+z^2+1)^2}{4x^2}. \tag{2.1.16}$$

From equation (2.1.14), we obtain

$$\frac{(\rho-1)(\mu-1)}{a-1} = \frac{(x^2+z^2-1)^2}{4x^2}. \tag{2.1.17}$$

We define

$$r = \left(\frac{\rho\mu}{a}\right)^{\frac{1}{2}} + \left(\frac{(\mu-1)(\rho-1)}{\mu-1}\right)^{\frac{1}{2}}, \tag{2.1.18}$$

then we substitute equation (2.1.16) and equation (2.1.17) in equation (2.1.18) to get

$$r = \frac{x^2+z^2+1}{2x} - \frac{x^2+z^2-1}{2x} = \frac{1}{x}$$

Hence,

$$x = \frac{1}{r} \tag{2.1.19}$$

Then plug equation (2.1.19) in equation (2.1.15), we have

$$z = \frac{1}{r}\left(\frac{(\mu-a)(a-\rho)}{a(a-1)}\right)^{\frac{1}{2}}.$$

Therefore, f is surjective. Suppose $f(\rho,\mu) = f(\bar{\rho},\bar{\mu}) = (x,z)$. By Proposition 1, $\rho,\mu,\bar{\rho},\bar{\mu}$ are all roots of $F(s) = 0$ with F defined in (2.1.12). Since $(\rho,\mu) \in A$, $(\bar{\rho},\ \bar{\mu}) \in A$, this implies $(\rho,\mu) = (\bar{\rho},\bar{\mu})$. Thus, f is injective. Hence, f is bijective. $\square$

If we change the signs of the square roots in (2.1.3) and (2.1.4) we obtain coordinates for other regions in $\mathbb{R}^2$ according to Table 1.

	$\left(\frac{\mu\rho}{a}\right)^{\frac{1}{2}}$	$\left(\frac{(\mu-1)(\rho-1)}{a-1}\right)^{\frac{1}{2}}$	$\left(\frac{(\mu-a)(a-\rho)}{a(a-1)}\right)^{\frac{1}{2}}$
$x > 0,\ z > 0,\ x^2 + z^2 < 1$	$+$	$+$	$+$
$x > 0,\ z > 0,\ x^2 + z^2 > 1$	$+$	$-$	$+$
$x > 0,\ z < 0,\ x^2 + z^2 > 1$	$+$	$-$	$-$
$x > 0,\ z < 0,\ x^2 + z^2 < 1$	$+$	$+$	$-$

Table 2.1: Signs of roots in regions of (x,z)-plane

Some coordinate lines are shown in Figure 2.1. Note that the boundary of the region is given by the quarter circle $\rho = 1$, the vertical segment $\mu = \infty$, and two horizontal segments $[0,b]$, $[b,1]$ represented by $\rho = a$ and $\mu = a$ respectively, where $b = \sqrt{a} - \sqrt{a-1} = (\sqrt{a} + \sqrt{a-1})^{-1}$.

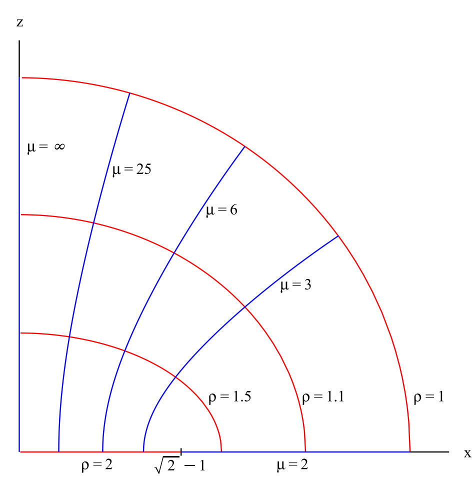

Figure 2.1: Coordinate lines of system (2.1.3) for $a = 2$

We calculate the metric coefficients in this chapter. We will use them in the later chapters.

Proposition 3. *The metric coefficients h_ρ and h_μ for (2.1.3) and (2.1.4) are given by*

$$\begin{aligned} h_\rho^2 &:= \left(\frac{\partial x}{\partial \rho}\right)^2 + \left(\frac{\partial z}{\partial \rho}\right)^2 \\ &= \frac{\mu - \rho}{4(a-\rho)\rho(\rho-1)}\frac{1}{r^2} \\ &= \frac{1}{16}\left(-\frac{(x^2+z^2+1)^2}{\rho^2} + \frac{(x^2+z^2-1)^2}{(\rho-1)^2} + \frac{4z^2}{(\rho-a)^2}\right) \end{aligned}$$

$$\begin{aligned} h_\mu^2 &:= \left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial z}{\partial \mu}\right)^2 \\ &= \frac{\mu - \rho}{4(\mu-a)\mu(\mu-1)}\frac{1}{r^2} \\ &= \frac{1}{16}\left(-\frac{(x^2+z^2+1)^2}{\mu^2} + \frac{(x^2+z^2-1)^2}{(\mu-1)^2} + \frac{4z^2}{(\mu-a)^2}\right) \end{aligned}$$

Proof. Let

$$J = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \mu} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \mu} \end{pmatrix},$$

where $(x, z) = f(\rho, \mu)$ according to (2.1.3) and (2.1.4). First, we wish to show that

$$J^T J = \begin{pmatrix} h_\rho^2 & 0 \\ 0 & h_\mu^2 \end{pmatrix},$$

where

$$\begin{aligned} h_\rho^2 &= \frac{\mu - \rho}{4(a-\rho)\rho(\rho-1)}\frac{1}{r^2}, \\ h_\mu^2 &= \frac{\mu - \rho}{4(\mu-a)\mu(\mu-1)}\frac{1}{r^2}. \end{aligned}$$

We calculate

$$\frac{\partial r}{\partial \rho} = \frac{1}{2}\left(\frac{\mu}{a\rho}\right)^{\frac{1}{2}} + \frac{1}{2}\left(\frac{\mu-1}{(a-1)(\rho-1)}\right)^{\frac{1}{2}}, \tag{2.1.20}$$

which can be used for the rest of this proposition. Let

$$L_1 = \frac{1}{r}\frac{\partial r}{\partial \rho}\left(1 + \frac{(\mu-a)(a-\rho)}{a(a-1)}\right). \tag{2.1.21}$$

We have

$$
\begin{aligned}
{h_\rho}^2 &= \left(-\frac{1}{r^2}\frac{\partial r}{\partial \rho}\right)^2 \\
&+ \left(-\frac{1}{r^2}\left(\frac{(\mu-a)(a-\rho)}{a(a-1)}\right)^{\frac{1}{2}}\left(\frac{\partial r}{\partial \rho}\right) - \frac{1}{2}\frac{1}{r}\left(\frac{(\mu-a)(a-\rho)}{a(a-1)}\right)^{-\frac{1}{2}}\left(\frac{\mu-a}{a(a-1)}\right)\right)^2 \\
&= \frac{1}{r^2}\left(\frac{1}{r^2}\left(\frac{\partial r}{\partial \rho}\right)^2\left(1+\frac{(\mu-a)(a-\rho)}{a(a-1)}\right) + \frac{1}{r}\left(\frac{\mu-a}{a(a-1)}\right)\left(\frac{\partial r}{\partial \rho}\right) + \frac{1}{4}\left(\frac{a(a-1)}{(\mu-a)(a-\rho)}\right)\left(\frac{\mu-a}{a(a-1)}\right)^2\right) \\
&= \frac{1}{r^2}\left(1+\frac{(\mu-a)(a-\rho)}{a(a-1)}\right)^{-1}\left(\left(L_1+\frac{1}{2}\left(\frac{\mu-a}{a(a-1)}\right)\right)^2 + \frac{1}{4}\left(\frac{\mu-a}{a(a-1)(a-\rho)}\right)\right) \\
&= \frac{\mu-\rho}{4(a-\rho)\rho(\rho-1)}\frac{1}{r^2}.
\end{aligned}
$$

Similarly, we have

$$h_\mu^2 = \frac{\mu-\rho}{4(\mu-a)\mu(\mu-1)}\frac{1}{r^2}.$$

Substitute (2.1.6) and (2.1.20) in (2.1.21) and simplify it then we have

$$L_1 = -\frac{1}{2}\frac{\mu-a}{a(a-1)} + \frac{1}{2}\left(\frac{\mu(\mu-1)}{a(a-1)}\right)^{\frac{1}{2}}\left(\left(\frac{\rho}{\rho-1}\right)^{\frac{1}{2}} - \left(\frac{\rho-1}{\rho}\right)^{\frac{1}{2}}\right).$$

Similarly, we have

$$L_2 := \frac{1}{r}\frac{\partial r}{\partial \mu}\left(1+\frac{(\mu-a)(a-\rho)}{a(a-1)}\right) = \frac{1}{2}\frac{a-\rho}{a(a-1)} + \frac{1}{2}\left(\frac{\rho(\rho-1)}{a(a-1)}\right)^{\frac{1}{2}}\left(\left(\frac{\mu}{\mu-1}\right)^{\frac{1}{2}} - \left(\frac{\mu-1}{\mu}\right)^{\frac{1}{2}}\right).$$

L_1 and L_2 will be used in the following calculation. The off-diagonal element in J^TJ multiplied by r^2 defined in (2.1.4) is

$$
\begin{aligned}
& r^2\left(\frac{\partial x}{\partial \rho}\frac{\partial x}{\partial \mu} + \frac{\partial z}{\partial \rho}\frac{\partial z}{\partial \mu}\right) = \frac{\frac{\partial r}{\partial \rho}}{r}\frac{\frac{\partial r}{\partial \mu}}{r} + \\
& \left(\frac{\frac{\partial r}{\partial \rho}}{-r}\left(\frac{(\mu-a)(a-\rho)}{a(a-1)}\right)^{\frac{1}{2}} - \frac{1}{2}\left(\frac{\mu-a}{(a-\rho)a(a-1)}\right)^{\frac{1}{2}}\right)\left(\frac{\frac{\partial r}{\partial \mu}}{-r}\left(\frac{(\mu-a)(a-\rho)}{a(a-1)}\right)^{\frac{1}{2}} + \frac{1}{2}\left(\frac{a-\rho}{(\mu-a)a(a-1)}\right)^{\frac{1}{2}}\right) \\
& = 0
\end{aligned}
$$

We can compute J^TJ in a slightly different way as follows. Let

$$K = \begin{pmatrix} \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial z} \\ \frac{\partial \mu}{\partial x} & \frac{\partial \mu}{\partial z} \end{pmatrix} = J^{-1}.$$

Then

$$J^T J = (KK^T)^{-1}.$$

Let

$$\begin{aligned}
F_s :&= \frac{(x^2+z^2+1)^2}{s} - \frac{(x^2+z^2-1)^2}{s-1} - \frac{4z^2}{s-a},\\
P_s :&= \frac{\partial F_s}{\partial s} = -\frac{(x^2+z^2+1)^2}{s^2} + \frac{(x^2+z^2-1)^2}{(s-1)^2} + \frac{4z^2}{(s-a)^2}\\
Q_s :&= \frac{x^2+z^2+1}{s} - \frac{x^2+z^2-1}{s-1}\\
\tilde{Q}_s :&= \frac{x^2+z^2+1}{s} - \frac{x^2+z^2-1}{s-1},\ \textit{where}\ s = \mu\ \textit{or}\ \rho.
\end{aligned}$$

In these definitions $x, z, \rho, and\ \mu$ are independent variables.
Let $(x, z) = f(\rho, \mu)$. We have

$$\begin{aligned}
\frac{\partial F_s}{\partial s}\frac{\partial s}{\partial x} &= P_\mu \frac{\partial s}{\partial x} = \frac{4x(x^2+z^2+1)}{s} - \frac{4x(x^2+z^2-1)}{s-1} = 4xQ_s\\
\frac{\partial F_s}{\partial s}\frac{\partial s}{\partial z} &= P_s \frac{\partial s}{\partial z} = \frac{4x(x^2+z^2+1)}{s} - \frac{4x(x^2+z^2-1)}{s-1} - 4z\frac{2}{s-a} = 4x(Q_s - \frac{2}{s-a}).
\end{aligned}$$

We can obtain

$$\begin{aligned}
P_s \frac{\partial s}{\partial x} &= 4xQ_s,\\
P_s \frac{\partial s}{\partial z} &= 4x(Q_s - \frac{2}{s-a}).
\end{aligned}$$

We have

$$\begin{aligned}
&\frac{P_s P_t}{16} k_{ij}\\
=&\frac{P_s P_t}{16}\left(\frac{\partial s}{\partial x}\frac{\partial t}{\partial x} + \frac{\partial s}{\partial z}\frac{\partial t}{\partial z}\right)\\
=&x^2 Q_s Q_t + z^2\left(Q_s - \frac{2}{s-a}\right)\left(Q_t - \frac{2}{t-a}\right)\\
=&\frac{1}{2}Q_s F_t + \frac{1}{2}Q_t F_s + \frac{F_t - F_s}{t-s}\ \textit{where}\ s \neq\ t,\ s = \rho\ \textit{or}\ \mu, \textit{and}\ t = \rho\ \textit{or}\ \mu.
\end{aligned}$$

Therefore,

$$x^2 Q_s Q_t + z^2\left(Q_s - \frac{2}{s-a}\right)\left(Q_t - \frac{2}{t-a}\right) = \frac{1}{2}Q_s F_t + \frac{1}{2}Q_t F_s + \frac{F_t - F_s}{t-s},\ \textit{if}\ s \neq t.$$

By taking limit $s \to t$, we obtain

$$x^2 {Q_t}^2 + z^2 \left(Q_t - \frac{2}{t-a} \right)^2 = Q_s F_t + P_t.$$

Therefore,

$$\frac{P_s P_t}{16} k_{ij} = x^2 Q_s Q_t + z^2 \left(Q_s - \frac{2}{s-a} \right) \left(Q_t - \frac{2}{t-a} \right) = \begin{cases} P_s, \text{if } s = t \\ 0, \text{ if } s \neq t. \end{cases}$$

Hence,

$$\begin{aligned} {h_\rho}^2 &= \frac{\mu - \rho}{4(a-\rho)\rho(\rho-1)} \frac{1}{R^2} = \frac{1}{16} \left(-\frac{(x^2+z^2+1)^2}{\rho^2} + \frac{(x^2+z^2-1)^2}{(\rho-1)^2} + \frac{4z^2}{(\rho-a)^2} \right) \\ {h_\mu}^2 &= \frac{\mu - \rho}{4(\mu-\rho)\mu(\mu-1)} \frac{1}{R^2} = \frac{1}{16} \left(-\frac{(x^2+z^2+1)^2}{\mu^2} + \frac{(x^2+z^2-1)^2}{(\mu-1)^2} + \frac{4z^2}{(\mu-a)^2} \right) \end{aligned}$$

□

2.2 Flat-Ring Coordinates In Transcendental Form

Miller [4] also considers the following transcendental form of flat-ring coordinates. We write $a = k^{-2}$ with $k \in (0, 1)$, and set

$$\rho = \mathrm{sn}^2(\beta, k), \quad \mu = \mathrm{sn}^2(\alpha, k), \tag{2.2.1}$$

where $\mathrm{sn}(z, k)$ denotes the Jacobian elliptic function of modulus k and z is a complex number. We will also use the Jacobian elliptic functions $\mathrm{cn}(z, k)$, $\mathrm{dn}(z, k)$, the complementary modulus $k' = \sqrt{1-k^2}$, and the complete elliptic integrals K, K'. K and K' are the complete elliptic integrals. More details about Jacobian elliptic functions are given in the appendix.

The interval $\beta \in (K - iK', K)$ is mapped bijectively onto the interval $\rho \in (1, a)$. The interval $\alpha \in (iK', K + iK')$ is mapped bijectively onto the interval $\mu \in (a, \infty)$. Substituting (2.2.1) into (2.1.1) and (2.1.2), we obtain

$$x = \frac{\cos\phi}{R}, \quad y = \frac{\sin\phi}{R}, \quad z = \frac{i\,\mathrm{dn}(\alpha, k)\mathrm{dn}(\beta, k)}{k' R}, \tag{2.2.2}$$

where

$$R = k\,\mathrm{sn}(\alpha, k)\mathrm{sn}(\beta, k) + \frac{k}{k'}\,\mathrm{cn}(\alpha, k)\mathrm{cn}(\beta, k).$$

The coordinate surface $\alpha = \alpha_0$ is given by

$$\frac{(x^2+y^2+z^2+1)^2}{\mathrm{sn}^2(\alpha_0,k)} + \frac{(x^2+y^2+z^2-1)^2}{\mathrm{cn}^2(\alpha_0,k)} + \frac{4k^2z^2}{\mathrm{dn}^2(\alpha_0,k)} = 0, \tag{2.2.3}$$

and the coordinate surface $\beta = \beta_0$ is given by

$$\frac{(x^2+y^2+z^2+1)^2}{\mathrm{sn}^2(\beta_0,k)} + \frac{(x^2+y^2+z^2-1)^2}{\mathrm{cn}^2(\beta_0,k)} + \frac{4k^2z^2}{\mathrm{dn}^2(\beta_0,k)} = 0. \tag{2.2.4}$$

The metric coefficients of (2.2.2) are

$$h_\alpha = h_{\beta'} = \frac{k(\mathrm{sn}^2(\alpha,k) - \mathrm{sn}^2(\beta,k))^{1/2}}{R}, \quad h_\phi = \frac{1}{R}, \tag{2.2.5}$$

where $\beta = K + i\beta'$ and β' is real.

When $\phi = 0$, we have the planar coordinate system

$$x = \frac{1}{R}, \quad z = \frac{i\,\mathrm{dn}(\alpha,k)\mathrm{dn}(\beta,k)}{k'R}. \tag{2.2.6}$$

The transcendental form of flat-ring coordinates has the advantage that it can be extended to a coordinate system for almost the whole space $\mathbb{R}^3$. We can achieve this in three different ways.

2.2.1 First variant of transcendental flat-ring coordinates

We take $\alpha \in (iK', K+iK')$ and $\beta \in (K-iK', K+3iK')$. Then it can be shown that (2.2.6) establishes a bijective real-analytic map between (α,β) and the region

$$Q_1 = \{(x,z) : x > 0\} \setminus \{(x,0) : 0 < x \le b^{-1} = \frac{1+k'}{k}\}.$$

Q_1 is the right-hand half-plane with a cut from $x = 0$ to $x = b^{-1}$ along the x-axis. The range of α, β in regions of the (x,z)-plane is shown in Table 2.2. Some coordinate lines of this coordinate system are depicted in Figure 2.2. The first variant of flat-ring coordinates will be useful when we discuss internal harmonics associated with coordinate surfaces $\alpha = \alpha_0$. When we discuss internal harmonics associated with coordinate surfaces $\beta = \beta_0$, then the following second variant will be more convenient.

	α	β
$x>0, z>0, x^2+z^2<1$	$(iK', K+iK')$	$(K-iK', K)$
$x>0, z>0, x^2+z^2>1$	$(iK', K+iK')$	$(K, K+iK')$
$x>0, z<0, x^2+z^2>1$	$(iK', K+iK')$	$(K+iK', K+2iK')$
$x>0, z<0, x^2+z^2<1$	$(iK', K+iK')$	$(K+2iK', K+3iK')$

Table 2.2: Range of α, β in regions of (x,z)-plane (first variant)

2.2.2 Second variant of transcendental flat-ring coordinates

Now we take $\alpha \in (iK', 2K+iK')$ and $\beta \in (K-iK', K+iK')$. Then (2.2.6) establishes a bijective real-analytic map between (α,β) and the region

$$Q_2 = \{(x,z) : x > 0\} \setminus \{(x,0) : 0 < x \leq b = \frac{k}{1+k'} \text{ or } x \geq b^{-1} = \frac{k}{1-k'}\}.$$

Q_2 is the right-hand half-plane with cuts along the x-axis from $x=0$ to $x=\frac{k}{1+k'}$ and from $\frac{k}{1-k'}$ to ∞ The range of α, β in regions of the (x,z)-plane is shown in Table 2.3.

	α	β
$x>0, z>0, x^2+z^2<1$	$(iK', K+iK')$	$(K-iK', K)$
$x>0, z>0, x^2+z^2>1$	$(iK', K+iK')$	$(K, K+iK')$
$x>0, z<0, x^2+z^2>1$	$(K+iK', 2K+iK')$	$(K, K+iK')$
$x>0, z<0, x^2+z^2<1$	$(K+iK', 2K+iK')$	$(K-iK', K)$

Table 2.3: Range of α, β in regions of (x,z)-plane (second variant)

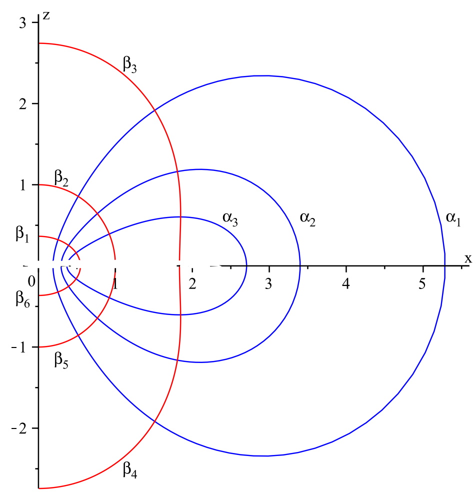

Figure 2.2: Coordinate lines $\alpha = \alpha_i$, $\alpha_1 = 0.3K + iK'$, $\alpha_2 = 0.5K + iK'$, $\alpha_3 = 0.7K + iK'$, and $\beta = \beta_i$, $\beta_1 = K - 0.5iK'$, $\beta_2 = K$, $\beta_3 = K + 0.5iK'$, $\beta_4 = K + 1.5iK'$, $\beta_5 = K + 2iK'$, $\beta_6 = K + 2.5iK'$ of system (2.2.6) for $a = 2$.

2.2.3 Third variant of transcendental flat-ring coordinates

We take $\alpha \in (iK', K+iK')$ and $\beta \in (K-3iK', K+iK')$. Then (2.2.6) establishes a bijective real-analytic map between (α, β) and the region

$$Q_3 = \{(x,z) : x > 0\} \setminus \{(x,0) : 0 < x \le b = \frac{k}{1+k'}\}.$$

2.2.4 Reflection

Let $x > 0, z > 0$ with coordinates $\alpha \in (iK', K+iK')$, $\beta \in (K-iK', K+iK')$ according to (2.2.6). These coordinates are the same in all three variants. Then the coordinates $\tilde{\alpha}$, $\tilde{\beta}$ of the point $(x, -z)$ are

$$\begin{array}{ll} \tilde{\alpha} = \alpha, \quad \tilde{\beta} = 2K + 2iK' - \beta & \text{for the first variant,} \\ \tilde{\alpha} = 2K + 2iK' - \alpha, \quad \tilde{\beta} = \beta & \text{for the second variant,} \\ \tilde{\alpha} = \alpha, \quad \tilde{\beta} = 2K - 2iK' - \beta & \text{for the third variant.} \end{array}$$

We see that the reflection $(x, z) \mapsto (x, -z)$ is represented by reflections in the coordinates but in each variant in a different way. We will use these reflections when we derive harmonic functions later.

Chapter 3

R-Separation of Variables

We wish to solve Laplace's equation in flat-ring coordinates by using R-separation of variables. We state one theorem that we will use in the last two chapters. Let $U(x, y, z)$ be a function defined in the region

$$\tilde{Q}_0 = \{(x\cos\phi, x\sin\phi, z) : (x, z) \in Q_0,\ \phi \in (-\pi, \pi)\}.$$

where

$$Q_0 = \{(x, z) : x > 0, z > 0, x^2 + z^2 < 1\}$$

In this region, we have flat-ring coordinates $\rho \in (1, a)$, $\mu \in (a, \infty)$, and $\phi \in (-\pi, \pi)$ as we showed in chapter 2.

Lemma 4. *Suppose*

$$U(x, y, z) = (x^2 + y^2)^{-1/4} u_1(\rho) u_2(\mu) u_3(\phi),$$

$\rho \in (1, a)$, $\mu \in (a, \infty)$, *and* $\phi \in (-\pi, \pi)$. *Assume* u_1, u_2, u_3 *are not identically zero. Then* U *is harmonic if and only if there are separation constants* m, h *such that* u_1 *and* u_2 *satisfy the algebraic Lamé equation*

$$\frac{d^2u}{ds^2} + \frac{1}{2}\left(\frac{1}{s} + \frac{1}{s-1} + \frac{1}{s-a}\right)\frac{du}{ds} + \frac{ah - (m^2 - \frac{1}{4})s}{4s(s-1)(s-a)}u = 0,$$

and u_3 *satisfies*

$$\frac{d^2u_3}{d\phi^2} + m^2 u_3 = 0.$$

Note that u_1 *and* u_2 *satisfy the same differential equation but on different intervals* $(1, a)$ *and* (a, ∞), *respectively.*

Proof. We consider cylindrical coordinates

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z.$$

Let $U(x, y, z) = u(r, z, \phi)$. Then the equation

$$\Delta U = 0$$

transforms to

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} = 0. \tag{3.0.1}$$

We wish to separate the variable ϕ from the other two variables r and z. We assume

$$u(r, z, \phi) = v(r, z)u_3(\phi). \tag{3.0.2}$$

We plug (3.0.2) in (3.0.1) then we have u_3 is a solution of

$$\frac{d^2 u_3}{d\phi^2} + m^2 u_3 = 0,$$

where m is a constant. It follows that

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2} = \frac{m^2}{r^2}v. \tag{3.0.3}$$

In order to separate the variables r and z, we set

$$v(r, z) = r^{-\frac{1}{2}}\tilde{v}(r, z).$$

We calculate

$$\frac{\partial v}{\partial r} = -\frac{1}{2}r^{\frac{-3}{2}}\ \tilde{v} + r^{\frac{-1}{2}}\frac{\partial \tilde{v}}{\partial r}, \tag{3.0.4}$$

and

$$\frac{\partial^2 v}{\partial r^2} = \frac{3}{4}r^{\frac{-5}{2}}\ \tilde{v} - r^{\frac{-3}{2}}\frac{\partial \tilde{v}}{\partial r} + r^{\frac{-1}{2}}\frac{\partial^2 \tilde{v}}{\partial r^2}. \tag{3.0.5}$$

From (3.0.4), we have

$$\frac{1}{2}r^{\frac{-3}{2}}\ \tilde{v} = -\frac{\partial v}{\partial r} + r^{\frac{-1}{2}}\frac{\partial \tilde{v}}{\partial r}. \tag{3.0.6}$$

Multiplying (3.0.6) both sides by $-\frac{3}{2r}$, we have

$$\frac{3}{4}r^{\frac{-5}{2}}\ \tilde{v} = -\frac{3}{2r}(\frac{\partial v}{\partial r} + r^{\frac{-1}{2}}\frac{\partial \tilde{v}}{\partial r}). \tag{3.0.7}$$

Substituting (3.0.7) in (3.0.5) and rearranging the terms, we obtain

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = \frac{1}{4} r^{\frac{-5}{2}} \tilde{v} + r^{\frac{-1}{2}} \frac{\partial^2 \tilde{v}}{\partial r^2}. \tag{3.0.8}$$

We substitute (3.0.8) in (3.0.3) then we have

$$\frac{\partial^2 \tilde{v}}{\partial r^2} + \frac{\partial^2 \tilde{v}}{\partial z^2} = \frac{m^2 - \frac{1}{4}}{r^2} \tilde{v}. \tag{3.0.9}$$

We change variables by setting

$$\tilde{v}(r, z) = w(\rho, \mu) \tag{3.0.10}$$

with flat-ring coordinates ρ and μ. The metric coefficients are

$$h_\rho = \frac{\sqrt{\mu - \rho}}{R f_1(\rho)}$$

and

$$h_\mu = \frac{\sqrt{\mu - \rho}}{R f_2(\mu)},$$

where

$$f_1(\rho) = 2\sqrt{(a - \rho)\rho(\rho - 1)},$$

$$f_2(\mu) = 2\sqrt{(\mu - a)\mu(\mu - 1)},$$

and

$$R = \left(\frac{\mu\rho}{a}\right)^{\frac{1}{2}} + \left(\frac{(\mu - 1)(\rho - 1)}{a - 1}\right)^{\frac{1}{2}}.$$

Equation (3.0.9) can be transformed to

$$\frac{1}{h_\rho h_\mu} \left[\frac{\partial}{\partial \rho} \left(\frac{h_\rho}{h_\mu} \frac{\partial w}{\partial \rho} \right) + \frac{\partial}{\partial \mu} \left(\frac{h_\mu}{h_\rho} \frac{\partial w}{\partial \mu} \right) \right] = R^2 \left(m^2 - \frac{1}{4} \right) w. \tag{3.0.11}$$

We plug h_ρ, h_μ, f_1, and f_2 from above in (3.0.11) and simplify it then we have

$$f_1(\rho) \frac{\partial}{\partial \rho} \left(f_1(\rho) \frac{\partial w}{\partial \rho} \right) + f_2(\mu) \frac{\partial}{\partial \mu} \left(f_2(\mu) \frac{\partial w}{\partial \mu} \right) = (m^2 - \frac{1}{4})(\mu - \rho) w. \tag{3.0.12}$$

Now we separate the variables ρ and μ by setting

$$w(\rho, \mu) = u_1(\rho) u_2(\mu). \tag{3.0.13}$$

We plug (3.0.13) in (3.0.12) and rearrange the terms and it follows that

$$\frac{f_1(\rho)\frac{\mathrm{d}}{\mathrm{d}\rho}\left(f_1(\rho)\frac{\mathrm{d}u_1}{\mathrm{d}\rho}\right)+(m^2-\frac{1}{4})\rho u_1}{u_1} \tag{3.0.14}$$

$$=\frac{-f_2(\mu)\frac{\mathrm{d}}{\mathrm{d}\mu}\left(f_2(\mu)\frac{\mathrm{d}u_2}{\mathrm{d}\mu}\right)+(m^2-\frac{1}{4})\mu u_2}{u_2}=h \tag{3.0.15}$$

where h is a constant. Then we have

$$\frac{d^2u_1}{d\rho^2}+\frac{1}{2}\left(\frac{1}{\rho}+\frac{1}{\rho-1}+\frac{1}{\rho-a}\right)\frac{du_1}{d\rho}+\frac{ah-(m^2-\frac{1}{4})\rho}{4\rho(\rho-1)(\rho-a)}u_1=0 \tag{3.0.16}$$

and

$$\frac{d^2u_2}{d\mu^2}+\frac{1}{2}\left(\frac{1}{\mu}+\frac{1}{\mu-1}+\frac{1}{\mu-a}\right)\frac{du_2}{d\mu}+\frac{ah-(m^2-\frac{1}{4})\mu}{4\mu(\mu-1)(\mu-a)}u_2=0. \tag{3.0.17}$$

□

Let $U(x,y,z)$ be a function defined in the region

$$\tilde{Q}_1=\{(x\cos\phi, x\sin\phi, z):(x,z)\in Q_1, \phi\in(-\pi,\pi)\},$$

where

$$Q_1=\{(x,z):x>0\}\setminus\{(x,0):0<x\le b^{-1}=\frac{1+k'}{k}\}.$$

Note that $\tilde{Q}_1$ consists of all of $\mathbb{R}^3$ with the exception of a disk in the xy-plane centered at the origin with radius b^{-1}, and the half-plane $y=0$, $x\le 0$. In this region, we have the first variant of transcendental flat-ring coordinates $\alpha\in(iK', K+iK')$, $\beta\in(K-iK', K+3iK')$, $\phi\in(-\pi,\pi)$.

Theorem 5. *Suppose that U has the form*

$$U(x,y,z)=(x^2+y^2)^{-1/4}v_1(\beta)v_2(\alpha)v_3(\phi), \tag{3.0.18}$$

where $\alpha\in(iK', K+iK')$, $\beta\in(K-iK', K+3iK')$, and $\phi\in(-\pi,\pi)$. Assume v_1, v_2, v_3 are not identically zero. Then U is harmonic if and only if there are separation constants m, h such that v_1 and v_2 both satisfy the transcendental Lamé equation

$$\frac{d^2v}{dz^2}+(h-(m^2-\tfrac{1}{4})k^2\mathrm{sn}^2(z,k))v=0, \tag{3.0.19}$$

and v_3 satisfies

$$\frac{d^2v_3}{d\phi^2}+m^2v_3=0. \tag{3.0.20}$$

Proof. It follows that v_3 satisfies

$$\frac{d^2v_3}{d\phi^2} + m^2 v_3 = 0 \tag{3.0.21}$$

from the above lemma. Substituting $\rho = sn^2(\beta, k)$, where $\beta \in (K-iK', K+3iK')$ in (3.0.16) and simplifying it, we have

$$\frac{d^2v}{dz^2} + (h - (m^2 - \tfrac{1}{4})k^2\mathrm{sn}^2(z,k))v = 0. \tag{3.0.22}$$

Similarly, substituting $\mu = sn^2(\alpha, k)$ where $\alpha \in (iK', K + iK')$ in (3.0.17) and simplifying it, we obtain

$$\frac{d^2v}{dz^2} + (h - (m^2 - \tfrac{1}{4})k^2\mathrm{sn}^2(z,k))v = 0. \tag{3.0.23}$$

□

Chapter 4

The Lamé Equation

4.1 The Lamé Equation

We collect some known results [13] on the Lamé equation

$$\frac{d^2w}{dz^2} + (h - \nu(\nu+1)k^2\mathrm{sn}^2(z,k))w = 0. \tag{4.1.1}$$

We assume that $0 < k < 1$ and $\nu \geq -\frac{1}{2}$. The function $\mathrm{sn}^2(z,k)$ is even and has period $2K$.

4.2 Lamé Periodic Fuctions

The problem we are going to consider in chapter 5 is to find values of the spectral parameter h such that (4.1.1) has nontrivial $4K$-periodic solutions. This eigenvalue problem is treated in [13, 15.5.1] and [21, Theorem 1.2]. The eigenvalue problem splits into four eigenvalue problems with separated boundary conditions

boundary conditions	eigenvalues	eigenfunctions	period
$w(0) = w(K) = 0$	$b_\nu^{2n+2}(k^2)$	$\mathrm{Es}_\nu^{2n+2}(z,k^2)$	$w(z+2K) = w(z)$
$w'(0) = w(K) = 0$	$b_\nu^{2n+1}(k^2)$	$\mathrm{Es}_\nu^{2n+1}(z,k^2)$	$w(z+2K) = -w(z)$
$w(0) = w'(K) = 0$	$a_\nu^{2n+1}(k^2)$	$\mathrm{Ec}_\nu^{2n+1}(z,k^2)$	$w(z+2K) = -w(z)$
$w'(0) = w'(K) = 0$	$a_\nu^{2n}(k^2)$	$\mathrm{Ec}_\nu^{2n}(z,k^2)$	$w(z+2K) = w(z)$

where $n \in \mathbb{N}_0$ in each case denotes the number of zeros of the eigenfunction in the interval $(0,K)$. We will normalize the eigenfunctions $w = \mathrm{Es}$ and $w = \mathrm{Ec}$ such that

$$\int_0^K w(t)^2\, dt = 1.$$

Since $\mathrm{sn}^2(z,k)$ also has the imaginary period $2iK'$, we can consider a second eigenvalue problem where h has to be determined in such a way that (4.1.1) admits nontrivial solutions with period $4iK'$. This problem can be reduced to the previous one by the substitutions

$$z' = i(z - K - iK'), \quad h' = \nu(\nu+1) - h.$$

Then we obtain

$$\frac{d^2w}{dz'^2} + (h' - \nu(\nu+1)k'^2\mathrm{sn}^2(z',k'))w = 0. \tag{4.2.1}$$

This is the same equation as (4.1.1) but with h', k' in place of h, k, respectively. Therefore, the $4iK'$-periodic solutions of (4.1.1) are

$$\mathrm{Ec'}_\nu^n(z,k^2) := \mathrm{Ec}_\nu^n(z',k'^2),$$
$$\mathrm{Es'}_\nu^{n+1}(z,k^2) := \mathrm{Es}_\nu^{n+1}(z',k'^2),$$

with corresponding eigenvalues

$$a'^n_\nu(k^2) = \nu(\nu+1) - a_\nu^n(k'^2), \quad b'^{n+1}_\nu(k^2) = \nu(\nu+1) - b_\nu^{n+1}(k'^2),$$

where $n \in \mathbb{N}_0$. $\mathrm{Ec'}_\nu^n$ have the period $2iK'$ if n is even, and $4iK'$ if n is odd. $\mathrm{Es'}_\nu^{n+1}$ have the period $2iK'$ if n is odd, and $4iK'$ if n is even.

4.3 Frobenius Solutions

The Lamé equation (4.1.1) has a regular singularity at $z = iK'$ with exponents $\nu+1, -\nu$. The Lamé function of the second kind $\mathrm{Fc}_\nu^n(z,k^2)$, $0 < \mathrm{Re}\, z < 2K$, is defined as the Frobenius solution of (4.1.1) with $h = a_\nu^n(k^2)$ belonging to the exponent $\nu+1$. It is known that Ec_ν^n and Fc_ν^n are linearly independent. Similarly, Fs_ν^{n+1} is the Frobenius solution of (3.0.19) with $h = b_\nu^{n+1}$ which belongs to the exponent $\nu+1$ at $z = iK'$. Similarly, we define $\mathrm{Fc'}_\nu^n$ and $\mathrm{Fs'}_\nu^{n+1}$.

We normalize the Lamé functions of the second kind such that

$$W[\mathrm{Ec}_\nu^n, \mathrm{Fc}_\nu^n] = W[\mathrm{Es}_\nu^{n+1}, \mathrm{Fs}_\nu^{n+1}] = 1, \tag{4.3.1}$$

where W is the Wronskian.

4.4 Lamé-Wangerin Functions

The problem we consider in chapter 6 is to find values of the spectral parameter h such that (4.1.1) has a nontrivial solution $w(z)$ that $sn(z,k)^{\frac{1}{2}}\, w(z)$ is bounded at iK' and $2K + iK'$.

The theory of this eigenvalue problem can be found in [13, 15.6] and [18, page 172].

boundary conditions	eigenvalues	eigenfunctions
$sn(z,k)^{\frac{1}{2}}w(z)$ *bounded at* $z = iK'$ *and* $z = 2K + iK'$	$c_\nu^n(k^2)$	$W_\nu^n(z,k^2)$

where $n \in \mathbb{N}_0$. $W_\nu^n(z,k^2)$ has exactly n zeros on the open interval $(iK', 2K+iK')$. $W_\nu^n(z,k^2)$ is an even function when n is even and $W_\nu^n(z,k^2)$ is an odd function when n is odd. We normalize the eigenfunctions $w = W_\nu^n$ such that

$$\int_0^{2K} w^2(t+iK')\,dt = 1.$$

It is more convenient to work on the Lamé equation by only considering real numbers. To do so, we can write

$$z = t + iK', \ \textit{where}\ 0 < t < 2K. \tag{4.4.1}$$

It is known that

$$sn(t+iK',k) = \frac{1}{k(sn(t,k))}. \tag{4.4.2}$$

Substituting (4.4.1) and (4.4.2) in (4.1.1), we have

$$\frac{d^2w}{dt^2} + \left(h - \frac{\nu(\nu+1)}{sn^2(t,k)}\right) w = 0, \ \textit{where}\ 0 < t < 2K. \tag{4.4.3}$$

More details about Lamé-Wangerin functions are given in the appendix.

Chapter 5

Flat-Ring Harmonic Functions

5.1 Internal Flat-Ring Harmonics

For internal flat-ring harmonics, we use the first variant of the flat-ring coordinates. We have $\alpha = t + iK'$ with $0 < t < K$. If $\alpha_0 = t_0 + iK'$ for some fixed value $t_0 \in (0, K)$, then $\alpha = \alpha_0$ describes a closed surface. The closed surface looks like a flat ring (see Figure 5.1). So we call this kind of closed surface "flat ring". Let

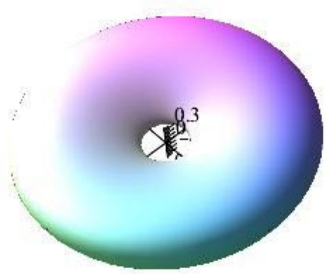

Figure 5.1: flat-ring, $a = 2$, $\alpha = 0.8K + iK'$

$$D_1 = \{\mathbf{r} \in \mathbb{R}^3 : \frac{(\|r\|^2 + 1)^2}{\mathrm{sn}^2(\alpha_0, k)} + \frac{(\|r\|^2 - 1)^2}{\mathrm{cn}^2(\alpha_0, k)} + \frac{4k^2 z^2}{\mathrm{dn}^2(\alpha_0, k)} > 0\}, \tag{5.1.1}$$

where $\alpha_0 \in (iK', K + iK')$. D_1 describes the interior of a flat-ring for a fixed α_0. Internal flat-ring harmonics are harmonic functions of the separated form (3.0.18) which are harmonic in the region described by D_1 for a fixed α_0. We can consider the region forms by the union of the regions described by D_1 for each α_0, which is all $\mathbb{R}^3$ except the z-axis. Suppose that v_1 and v_2 are solutions of the Lamé equation (3.0.19) defined on the strip $0 < Real\ v_s < 2K$, where $s = 1, 2$, and v_3 is a solution of (3.0.20). We use the first variant of the transcendental flat-ring coordinate system and $-\pi < \phi < \pi$ then $U(x, y, z)$ defined by (3.0.18) is a harmonic function in $\tilde{Q}_1$. We want this function to be harmonic on $\mathbb{R}^3$ except the z-axis. Clearly, we need $m \in \mathbb{Z}$, and we choose $u_3(\phi) = e^{im\phi}$ (alternatively, we could use $\cos(m\phi)$, $m = 0, 1, 2, \ldots$ and $\sin(m\phi)$, $m = 1, 2, 3, \ldots$). Then we have to require that the function $v_1(\beta)v_2(\alpha)$ is analytic in the right-hand half plane $x > 0, z \in \mathbb{R}$. We know $v_1(\beta)v_2(\alpha)$ is always analytic in the quadrant $x > 0, z > 0$. When we use the third variant, we can analytically extend this function to the quadrant $x > 0, z < 0$ across the segment $b = \frac{k}{1-k'} > x > 0$, $z = 0$. We want the first and third extension to be the same. In order to make the first and third extension the same, we need that v_1 to be periodic with period $4iK'$. Therefore, we take $v_1 = \mathrm{Ec}'^{n}_{|m|-\frac{1}{2}}$ or $v_1 = \mathrm{Es}'^{n+1}_{|m|-\frac{1}{2}}$. When we use the second variant, we can analytically extend this function to the quadrant $x > 0, z < 0$ across the segment $b = \frac{k}{1+k'} < x < b^{-1} = \frac{k}{1-k'}$, $z = 0$. We want the first and second extension to be the same. In order to make the first and second extension the same, we need that v_1 has the same parity with respect to $K + iK'$ as v_2. Therefore, v_1 has to be a constant multiple of v_2. We can take $v_1 = v_2$. We define internal flat-ring harmonics by

$$\mathrm{Gc}^n_m(x, y, z) = (x^2 + y^2)^{-1/4} \mathrm{Ec}'^{n}_{|m|-\frac{1}{2}}(\beta, k^2) \mathrm{Ec}'^{n}_{|m|-\frac{1}{2}}(\alpha, k^2) e^{im\phi}, \tag{5.1.2}$$

$$\mathrm{Gs}^{n+1}_m(x, y, z) = (x^2 + y^2)^{-1/4} \mathrm{Es}'^{n+1}_{|m|-\frac{1}{2}}(\beta, k^2) \mathrm{Es}'^{n+1}_{|m|-\frac{1}{2}}(\alpha, k^2) e^{im\phi}, \tag{5.1.3}$$

where $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$. Note that the Lamé functions $\mathrm{Ec}'(\zeta)$ and $\mathrm{Es}'(\zeta)$ are analytic on the strip $0 < \mathrm{Re}\,\zeta < 2K$, and that α and β lie in this strip.

We collect some properties of internal harmonics in the following theorem. We refer to the inversion at the unit sphere given by

$$\sigma(\mathbf{r}) = \|\mathbf{r}\|^{-2}\mathbf{r}.$$

Theorem 6. *The internal flat-ring harmonics* Gc^n_m *and* Gs^{n+1}_m *are harmonic in all of* $\mathbb{R}^3$ *except for the z-axis. They have the following symmetry regarding inversion at the unit sphere:*

$$\mathrm{Gc}^n_m(\sigma(\mathbf{r})) = \|\mathbf{r}\| \mathrm{Gc}^n_m(\mathbf{r}),$$
$$\mathrm{Gs}^{n+1}_m(\sigma(\mathbf{r})) = -\|\mathbf{r}\| \mathrm{Gs}^{n+1}_m(\mathbf{r}),$$

and regarding reflection at the xy-plane

$$\begin{aligned}\mathrm{Gc}_m^n(x,y,-z) &= (-1)^{n+1}\mathrm{Gc}_m^n(x,y,z),\\ \mathrm{Gs}_m^{n+1}(x,y,-z) &= (-1)^n\mathrm{Gs}_m^{n+1}(x,y,z).\end{aligned}$$

Proof. From our consideration at the beginning of this section we know that $U = \mathrm{Gc}$ and $U = \mathrm{Gs}$ are harmonic functions on $\mathbb{R}^3$ except for the z-axis and the two circles centered at the origin with radii b^{-1} and b on the xy-plane, respectively. The function U is bounded in a neighborhood of the two circles. Therefore, the two circles are removable singularities of U (see [3], Theorem XIII, page 271). Hence, U is harmonic on $\mathbb{R}^3$ except for the z-axis.

The inversion σ in the upper half-space $z > 0$ is expressed in flat-ring coordinates by the reflection of β at K: $\beta \mapsto 2K - \beta$. The Lamé functions Ec' remain unchanged under this reflection while the Lamé functions Es' change sign. Similarly, the reflection $z \mapsto -z$ is expressed by reflection of β at $K + iK'$: $\beta \mapsto 2(K + iK') - \beta$. The Lame functions Ec'^n_ν, Es'^n_ν remain unchanged under this reflection if n is odd but change sign when n is even. □

5.2 The Dirichlet Problem

From the known orthogonality and completeness properties of the periodic Lamé functions, we obtain the following theorem.

Theorem 7. *The system of functions*

$$(8\pi)^{-1/2}\mathrm{Ec}'^{\,n}_{|m|-\frac{1}{2}}(\beta,k^2)e^{im\phi},\ (8\pi)^{-1/2}\mathrm{Es}'^{\,n+1}_{|m|-\frac{1}{2}}(\beta,k^2)e^{im\phi},\quad m\in\mathbb{Z}, n\in\mathbb{N}_0$$

is an orthonormal basis in the Hilbert space

$$H_1 = L^2((K - iK', K + 3iK') \times (-\pi,\pi)).$$

We use the internal flat-ring harmonics to solve the Dirichlet problem for harmonic functions in the region D_1 defined in (5.1.1).

Theorem 8. *Let f be a function defined on the boundary ∂D_1 of the region D_1. Suppose that f is represented in flat-ring coordinates as*

$$(x^2+y^2)^{1/4}f(x,y,z) = g(\beta,\phi),\quad \beta = K + i\beta',\ \beta'\in(-K,3K), \phi\in(-\pi,\pi)$$

such that $g \in H_1$. For all $m \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, define

$$\begin{aligned}c_m^n :&= \frac{1}{8\pi\mathrm{Ec}'^n_{|m|-\frac{1}{2}}(\alpha_0,k^2)}\int_{-\pi}^{\pi}\int_{-K}^{3K} g(\beta,\phi)\mathrm{Ec}'^n_{|m|-\frac{1}{2}}(\beta,k^2)e^{-im\phi}\,d\beta' d\phi\\ &= \frac{1}{8\pi\left\{\mathrm{Ec}'^n_{|m|-\frac{1}{2}}(\alpha_0,k^2)\right\}^2}\int_{\partial D_1}\frac{1}{h_{\beta'}(\mathbf{r})}f(\mathbf{r})\mathrm{Gc}^n_{-m}(\mathbf{r})\,dS(\mathbf{r})\end{aligned}$$

and

$$d_m^{n+1} := \frac{1}{8\pi \mathrm{Es'}^{n+1}_{|m|-\frac{1}{2}}(\alpha_0,k^2)} \int_{-\pi}^{\pi}\int_{-K}^{3K} g(\beta,\phi)\mathrm{Es'}^{n+1}_{|m|-\frac{1}{2}}(\beta,k^2)e^{-im\phi}\,d\beta' d\phi$$
$$= \frac{1}{8\pi\left\{\mathrm{Ec'}^{n+1}_{|m|-\frac{1}{2}}(\alpha_0,k^2)\right\}^2}\int_{\partial D_1}\frac{1}{h_{\beta'}(\mathbf{r})}f(\mathbf{r})\mathrm{Gs}^{n+1}_{-m}(\mathbf{r})\,dS(\mathbf{r}).$$

Then the function

$$u(\mathbf{r}) = \sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty}\left(c_m^n \mathrm{Gc}_m^n(\mathbf{r}) + d_m^{n+1}\mathrm{Gs}_m^{n+1}(\mathbf{r})\right) \tag{5.2.1}$$

is harmonic in D_1 and it assumes the boundary values f on ∂D_1 in the weak sense. The infinite series in (5.2.1) *converges absolutely and uniformly in compact subsets of D_1.*

Proof. Using flat-ring coordinates, we can write surface integrals over ∂D_1 as double integrals:

$$\int_{\partial D_1} f(\mathbf{r})\,dS(\mathbf{r}) = \int_{-K}^{3K}\int_{-\pi}^{\pi} h_{\beta'}h_\phi(x^2+y^2)^{-1/4}g(\beta,\phi)\,d\beta'\,d\phi$$
$$= \int_{-K}^{3K}\int_{-\pi}^{\pi} h_{\beta'}(x^2+y^2)^{1/4}g(\beta,\phi)\,d\beta'\,d\phi.$$

with the metric coefficients $h_{\beta'}, h_\phi$ given in (2.2.5). This shows that the two formulas given for c_m^n and d_m^{n+1} agree.

The rest of the proof is similar to the proof of Theorem 6.3 in [6] . □

5.3 External Flat-Ring Harmonics

External flat-ring harmonics are harmonic functions U of the form (3.0.18) which are harmonic outside of all flat rings (5.1.1). Therefore, they are harmonic in $\mathbb{R}^3$ except for the annulus $b^2 \le x^2+y^2 \le b^{-2}$ in the (x,y)-plane. It is clear that m must be an integer and arguing as at the beginning of Section 5.1, we see that v_1 must have period $4iK'$. Since the function U has to be analytic along the z-axis, we require that v_2 is a solution of the Lamé equation (3.0.19) which is bounded (actually must converge to 0) when α approached iK'. These are the Frobenius solutions of the Lamé equation that we mentioned in Section 4.2. Thus we define external flat-ring harmonics by

$$\mathrm{Hc}_m^n(x,y,z) = (x^2+y^2)^{-1/4}\mathrm{Ec'}^n_{|m|-\frac{1}{2}}(\beta,k^2)\mathrm{Fc'}^n_{|m|-\frac{1}{2}}(\alpha,k^2)e^{im\phi}, \tag{5.3.1}$$
$$\mathrm{Hs}_m^{n+1}(x,y,z) = (x^2+y^2)^{-1/4}\mathrm{Es'}^{n+1}_{|m|-\frac{1}{2}}(\beta,k^2)\mathrm{Fs'}^{n+1}_{|m|-\frac{1}{2}}(\alpha,k^2)e^{im\phi}, \tag{5.3.2}$$

where $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$. We use the first variant of the transcendental flat-ring coordinates (we can also use the third variant but not the second one).

Theorem 9. *The external flat-ring harmonics* $\mathrm{Hc}_m^n, \mathrm{Hs}_m^{n+1}$ *are harmonic in all of* $\mathbb{R}^3$ *except for the closed annulus in the xy-plane centered at the origin with inner radius* $\frac{k}{1+k'}$ *and outer radius* $\frac{k}{1-k'}$. *They have the same symmetry properties as the internal flat-ring harmonics; see Theorem 6.*

Proof. From our consideration at the beginning of this chapter we know that the functions $U = \mathrm{Hc}_m^n$ and $U = \mathrm{Hs}_m^{n+1}$ are harmonic on $\mathbb{R}^3$ except for the z-axis and the annulus $b^2 \le x^2 + y^2 \le b^{-2}$ in the (x, y)-plane. Since U is bounded the z-axis is a removable singularity (see [3], Theorem VI, page 335). Therefore, U is harmonic on $\mathbb{R}^3$ except for the annulus $b^2 \le x^2 + y^2 \le b^{-2}$ in the (x, y)-plane. (It appears that U has a continuous extension on this annulus but not an analytic extension.) $\square$

We now show that external harmonics admit an integral representation in terms of internal harmonics.

Theorem 10. *Let* $\alpha_0 \in (iK', K + iK')$, $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$, *and let* $\mathbf{r}'$ *be a point outside* $\bar{D}_1$, *where* D_1 *is given by* (5.1.1). *Then*

$$\mathrm{Hc}_m^n(\mathbf{r}') = \frac{1}{\{\mathrm{Ec}'^n_{|m|-\frac{1}{2}}(\alpha_0, k^2)\}^2} \int_{\partial D_1} \frac{\mathrm{Gc}_m^n(\mathbf{r})}{h_\alpha(\mathbf{r}) 4\pi \|\mathbf{r} - \mathbf{r}'\|} \, dS(\mathbf{r}), \tag{5.3.3}$$

and

$$\mathrm{Hs}_m^{n+1}(\mathbf{r}') = \frac{1}{\{\mathrm{Es}'^{n+1}_{|m|-\frac{1}{2}}(\alpha_0, k^2)\}^2} \int_{\partial D_1} \frac{\mathrm{Gs}_m^{n+1}(\mathbf{r})}{h_\alpha(\mathbf{r}) 4\pi \|\mathbf{r} - \mathbf{r}'\|} \, dS(\mathbf{r}). \tag{5.3.4}$$

Proof. Let D be an open bounded subset of $\mathbb{R}^3$ with smooth boundary. For $u, v \in C^2(\bar{D})$, Green's formula states that

$$\int_D (u\Delta v - v\Delta u) \, d\mathbf{r} = \int_{\partial D} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) \, dS, \tag{5.3.5}$$

where $\frac{\partial u}{\partial \nu}$ is the outward normal derivative of u on the boundary ∂D of D. We apply (5.3.5) to $D = D_1$, $u = G = \mathrm{Gc}_m^n$, and

$$v(\mathbf{r}) = \frac{1}{4\pi \|\mathbf{r}' - \mathbf{r}\|}. \tag{5.3.6}$$

Since $\Delta u = \Delta v = 0$ on D_1 we obtain

$$0 = \int_{\partial D_1} \left(G \frac{\partial v}{\partial \nu} - v \frac{\partial G}{\partial \nu} \right) dS. \tag{5.3.7}$$

Let $B_r(\mathbf{q})$ denote the open ball centered at $\mathbf{q} \in \mathbb{R}^3$ with radius $r > 0$. We apply (5.3.5) a second time with $D = B_R(\mathbf{0}) - \bar{D}_1 - B_\epsilon(\mathbf{r}')$ with large R and small $\epsilon > 0$. Choose $u = H = \mathrm{Hc}_m^n$ and v as in (5.3.6). Note that $\Delta u = \Delta v = 0$ on D. By a standard argument, taking the limit $\epsilon \to 0$, we obtain

$$H(\mathbf{r}') = \int_{\partial B_R(\mathbf{0})} \left(H\frac{\partial v}{\partial \nu} - v\frac{\partial H}{\partial \nu} \right) dS - \int_{\partial D_1} \left(H\frac{\partial v}{\partial \nu} - v\frac{\partial H}{\partial \nu} \right) dS, \tag{5.3.8}$$

where, in the second integral, $\frac{\partial}{\partial \nu}$ denotes again the derivative in the direction of the outward normal as in (5.3.7). As $R \to \infty$, the first integral on the right-hand side of (5.3.8) tends to 0 (see [8], page 109), so we obtain

$$H(\mathbf{r}') = -\int_{\partial D_1} \left(H\frac{\partial v}{\partial \nu} - v\frac{\partial H}{\partial \nu} \right) dS, \tag{5.3.9}$$

Set $E(\zeta) = \mathrm{Ec}'^n_{|m|-\frac{1}{2}}(\zeta)$ and $F(\zeta) = \mathrm{Fc}'^n_{|m|-\frac{1}{2}}(\zeta)$. We multiply (5.3.7) by $F(\alpha_0)$, then multiply (5.3.9) by $E(\alpha_0)$ and add these equations. Using the definitions (5.1.2) and (5.3.1) of internal and external flat-ring harmonics and cancelling terms, we find

$$E(\alpha_0)H(\mathbf{r}') = \int_{\partial D_1} v \left(E(\alpha_0)\frac{\partial H}{\partial \nu} - F(\alpha_0)\frac{\partial G}{\partial \nu} \right) dS. \tag{5.3.10}$$

Since flat-ring coordinates are orthogonal, the normal derivative and the partial derivative with respect to α are related by

$$\frac{\partial}{\partial \nu} = \frac{1}{h_\alpha}\frac{\partial}{\partial \alpha},$$

where h_α is given in (2.2.5). Let $\mathbf{r} = (x, y, z) \in \partial D_1$ with flat-ring coordinates α, β, ϕ. Then

$$\begin{aligned}
&E(\alpha_0)\frac{\partial H}{\partial \nu}(\mathbf{r}) - F(\alpha_0)\frac{\partial G}{\partial \nu}(\mathbf{r}) \\
&\quad = E(\alpha_0)\frac{\partial((x^2+y^2)^{-1/4})}{\partial \nu}F(\alpha_0)E(\beta)e^{im\phi} \\
&\qquad +E(\alpha_0)(x^2+y^2)^{-1/4}h_\alpha^{-1}F'(\alpha_0)E(\beta)e^{im\phi} \\
&\qquad -F(\alpha_0)\frac{\partial((x^2+y^2)^{-1/4})}{\partial \nu}E(\alpha_0)E(\beta)e^{im\phi} \\
&\qquad -F(\alpha_0)(x^2+y^2)^{-1/4}h_\alpha^{-1}E'(\alpha_0)E(\beta)e^{im\phi} \\
&\quad = h_\alpha^{-1}(x^2+y^2)^{-1/4}\left\{E(\alpha_0)F'(\alpha_0) - E'(\alpha_0)F(\alpha_0)\right\}E(\beta)e^{im\phi}.
\end{aligned}$$

We now use the Wronskian (4.3.1) and obtain

$$E(\alpha_0)\frac{\partial H}{\partial \nu}(\mathbf{r}) - F(\alpha_0)\frac{\partial G}{\partial \nu}(\mathbf{r}) = \frac{1}{h_\alpha(\mathbf{r})E(\alpha_0)}G(\mathbf{r}). \tag{5.3.11}$$

When we substitute (5.3.11) in (5.3.10) we obtain (5.3.3). The proof of (5.3.4) is similar. □

5.4 A Fundamental Solution

We obtain the expansion of (5.3.6) in internal and external flat-ring harmonics of the first kind by combining Theorems 8 and 10.

Theorem 11. *Let* $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^3$ *with flat-ring coordinates* $\alpha, \alpha' \in (iK', K + iK')$, *respectively. If* $\operatorname{Re}\alpha > \operatorname{Re}\alpha'$ *then*

$$\frac{1}{\|\mathbf{r}-\mathbf{r}'\|} = \frac{1}{2}\sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty}\left(\mathrm{Gc}_m^n(\mathbf{r})\mathrm{Hc}_{-m}^n(\mathbf{r}') + \mathrm{Gs}_m^{n+1}(\mathbf{r})\mathrm{Hs}_{-m}^{n+1}(\mathbf{r}')\right). \tag{5.4.1}$$

Proof. We pick $\alpha_0 \in (iK', K+iK')$ such that $\operatorname{Re}\alpha' < \operatorname{Re}\alpha_0 < \operatorname{Re}\alpha$, and consider the domain D_1 defined in (5.1.1). The function $f(\mathbf{q}) := \frac{1}{\|\mathbf{q}-\mathbf{r}'\|}$ is harmonic on an open set containing $\bar{D}_1$. Therefore, by Theorem 8, we have

$$f(\mathbf{r}) = \sum_{m\in\mathbb{Z}}\sum_{n=0}^{\infty}\left(c_m^n \mathrm{Gc}_m^n(\mathbf{r}) + d_m^{n+1}\mathrm{Gs}_m^{n+1}(\mathbf{r})\right), \tag{5.4.2}$$

where c_m^n and d_m^{n+1} can be evaluated by Theorem 10:

$$c_n^m = \frac{1}{2}\mathrm{Hc}_{-m}^n(\mathbf{r}'), \quad d_m^{n+1} = \frac{1}{2}\mathrm{Hs}_{-m}^{n+1}(\mathbf{r}').$$

We use $h_{\beta'} = h_\alpha$. Thus we obtain (5.4.1). $\square$

Chapter 6

Peanut Harmonic Functions

6.1 External Peanut Harmonics

It is more convenient to consider external peanut harmonics first. We use the second variant of the flat-ring coordinates. We have $\beta = K + it$ with $-K' < t < K'$. If $\beta_0 = K + it_0$ for some fixed value $t_0 \in (-K', K')$, then $\beta = \beta_0$ describes a closed surface (if we add two points on the z-axis, where the coordinate system is not valid). The closed surface looks like a peanut (see Figure 6.1). So we call this kind of closed surface "peanut". The interior of

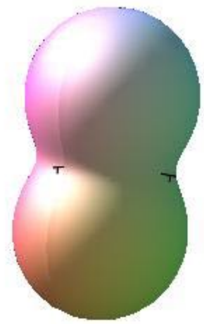

Figure 6.1: peanut, $a = 2, \beta = K + 0.7iK'$

a peanut is given by $t < t_0$ and the exterior by $t > t_0$ (if we add a disk or the exterior of a disk in the (x, y)-plane, where the coordinate system is not valid).

We also want to describe the region D_2(one peanut) in Cartesian coordinates as we did for the region D_1(one flat ring) earlier. We consider the surface given by

$$f(\mathbf{r}) := \frac{(\|\mathbf{r}\|^2+1)^2}{\mathrm{sn}^2(\beta_0,k)} + \frac{(\|\mathbf{r}\|^2-1)^2}{\mathrm{cn}^2(\beta_0,k)} + \frac{4k^2z^2}{\mathrm{dn}^2(\beta_0,k)} = 0. \tag{6.1.1}$$

Then $f(\sigma(\mathbf{r})) = \|\mathbf{r}\|^{-4} f(\mathbf{r})$, where $\sigma(\mathbf{r}) = \|\mathbf{r}\|^{-2}\mathbf{r}$ denote an inversion at the unit sphere. It follows that if $\mathbf{r}$ is on the surface, then the mirror point $\sigma(\mathbf{r})$ is on the surface. Therefore, the surface given by (6.1.1) has two parts, namely the surface $\beta = \beta_0$ and $\beta = \bar{\beta}_0$. Unless $t_0 = 0$, one surface is interior to the unit sphere and the other one is its mirror image and it lies exterior to the unit sphere. This is also clear from the observation that $f(\mathbf{r})$ remains the same if we replace β_0 by its conjugate. Between the surfaces we have $f(\mathbf{r}) > 0$ and interior to the smaller surface and exterior to the larger surface we have $f(\mathbf{r}) < 0$. We can check this by considering $\mathbf{r} = 0$ and $\|\mathbf{r}\| = 1$.

We consider three cases:

- If $t_0 = 0$ then the surface $\beta = \beta_0$ is the unit sphere and its interior D_2 is the open unit ball
$$B = \{\mathbf{r} : \|r\| < 1\}.$$
Notice that $f(\mathbf{r})$ is not well-defined in this case.

- If $-K' < t_0 < 0$ then the surface $\beta = \beta_0$ is in B. Then we have
$$D_2 = B \cap \{\mathbf{r} : f(\mathbf{r}) < 0\}.$$

- If $0 < t_0 < K'$ then the surface $\beta = \beta_0$ is exterior to the unit sphere and
$$D_2 = B \cup \{\mathbf{r} : f(\mathbf{r}) > 0\}.$$

External peanut harmonics are harmonic functions U of the form (3.0.18) that are harmonic outside a peanut described by D_2 for a fixed β. We can consider the region formed by the union of the regions outside each peanut described by D_2, which is $\mathbb{R}^3$ except the disc centered at the origin with radius $b = \frac{k}{1+k'}$ on the xy-plane. Suppose that v_1 and v_2 are solutions of the Lamé equation (3.0.19) defined on the strip $0 < \mathit{Real}\ v_s < 2K$, where $s = 1, 2$, and v_3 is a solution of (3.0.20). We use the second variant of the transcendental flat-ring coordinate system and $-\pi < \phi < \pi$ then $U(x, y, z)$ defined by (3.0.18) is a harmonic function in $\tilde{Q}_2$. We want this function to be harmonic on $\mathbb{R}^3$ except for the disc centered at the origin with radius $b = \frac{k}{1+k'}$ on the xy-plane. Clearly, we need $m \in \mathbb{Z}$ and we choose $u_3(\phi) = e^{im\phi}$ (alternatively, we could use $\cos(m\phi)$, $m = 0, 1, 2, \dots$ and $\sin(m\phi)$, $m = 1, 2, 3, \dots$). Then we have to require that the function $v_1(\beta)v_2(\alpha)$ is analytic in the right-hand half plane $x > 0, z \in \mathbb{R}$ except the segment between the origin and b on the x-axis. As harmonic functions are smooth, the function v_2 has to be bounded at the points iK' and $2K + iK'$. Therefore, we take $v_2 = W^n_{|m|-\frac{1}{2}}$. We know $v_1(\beta)v_2(\alpha)$ is always analytic in the quadrant $x > 0, z > 0$. When we use the first variant, we can analytically extend this function to the quadrant $x > 0, z < 0$ across the ray $x > b^{-1} = \frac{k}{1-k'}$, $z = 0$. When we use the second variant, we can analytically extend this function to the quadrant $x > 0, z < 0$ across the segment $b^{-1} > x > b, z = 0$. We want the first and second extension to be the same. In order to make the first and second extension the same, we need v_1 to have the same parity with respect to $K + iK'$ as v_2. Therefore, v_1 has to be a constant multiple of v_2. We can take $v_1 = v_2$. Thus we define external peanut harmonics by

$$H^n_m(x, y, z) = (x^2 + y^2)^{-1/4} W^n_{|m|-\frac{1}{2}}(\beta, k^2) W^n_{|m|-\frac{1}{2}}(\alpha, k^2) e^{im\phi}. \tag{6.1.2}$$

We collect some properties of external peanut harmonics in the following theorem.

Theorem 12. *The external peanut harmonics H^n_m are harmonic on $\mathbb{R}^3$ except for the disc centered at the origin with radius $b = \frac{k}{1+k'}$ on the xy-plane. Regarding reflection at the xy-plane, we have*

$$H^n_m(x, y, -z) = (-1)^n H^n_m(x, y, z).$$

Proof. From our consideration at the beginning of this section we know that H^n_m is a harmonic function on $\mathbb{R}^3$ except for the disc centered at the origin with the radius b on the xy-plane, z-axis, and the circle centered at the origin with radius b^{-1} on the xy-plane. H^n_m is bounded in a neighborhood of the circle. Therefore, the circle is a removable singularity of H^n_m (see [3], Theorem XIII, page 271). Since H^n_m is bounded in a neighborhood of z-axis, the z-axis is a removable singularity (see [3], Theorem VI, page 335). Hence, H^n_m are harmonic on $\mathbb{R}^3$ except for the disc centered at the origin with radius $b = \frac{k}{1+k'}$ on the xy-plane.

The reflection $z \mapsto -z$ is expressed by reflection of α at $K + iK'$: $\alpha \mapsto 2(K + iK') - \alpha$. The Lamé-Wangerin function W^n_ν remains unchanged under this reflection if n is even but changes sign if n is odd. □

6.2 Internal Peanut Harmonics

Internal peanut harmonics are harmonic functions U of the form (3.0.18) which are harmonic inside a peanut described by D_2 for a fixed β. We can consider the region formed by the union of the regions inside each peanut described by D_2, which is $\mathbb{R}^3$ except for the infinite annulus with inner radius $b = \frac{k}{1-k'}$ and outer radius ∞ in the (x, y)-plane. It is clear that m must be an integer and arguing as at the beginning of section 6.1.

We define internal peanut harmonics by

$$G^n_m(x, y, z) = (x^2 + y^2)^{-1/4} W^n_{|m|-\frac{1}{2}}(2K - \beta, k^2) W^n_{|m|-\frac{1}{2}}(\alpha, k^2) e^{im\phi}, \tag{6.2.1}$$

where $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$.

Theorem 13. *The internal peanut harmonics G_m^n are harmonic on $\mathbb{R}^3$ except for the infinite annulus with inner radius $b^{-1} = \frac{k}{1-k'}$ and outer radius ∞ in the (x,y)-plane. They have the following symmetry regarding inversion at the unit sphere (Kelvin inversion):*

$$G_m^n(\sigma(\mathbf{r})) = ||\mathbf{r}|| H_m^n(\mathbf{r}),$$

and regarding reflection at the xy-plane

$$G_m^n(x, y, -z) = (-1)^n G_m^n(x, y, z).$$

Proof. The inversion σ for $x > 0$ is expressed in the flat-ring coordinates by the reflection of β at K: $\beta \to 2K - \beta$. The region of all internal peanuts is under the inversion σ of D_2 for all β_0.

The reflection $z \mapsto -z$ is expressed by reflection of α at $K + iK'$: $\alpha \mapsto 2(K + iK') - \alpha$. The Lamé-Wangerin function W_ν^n remains unchanged under this reflection if n is even but changes sign if n is odd. □

We now show that external harmonics admit an integral representation in terms of internal harmonics. Assume

$$W[E_\nu^n, F_\nu^n](z) = w, \tag{6.2.2}$$

where W is the Wronskian, $E_\nu^n(z) = W_\nu^n(z)$, $F_\nu^n(z) = W_\nu^n(2K - z)$, and w is a constant.

Theorem 14. *Let $\beta_0 \in (K - iK', K + iK')$, $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$, and let $\mathbf{r}'$ be a point outside $\bar{D}_2$, where D_2 is given in section 6.1. Then*

$$H_m^n(\mathbf{r}') = \frac{w}{\{W_{|m|-\frac{1}{2}}^n(\beta_0, k^2)\}^2} \int_{\partial D_2} \frac{G_m^n(\mathbf{r})}{h_\beta(\mathbf{r}) 4\pi \|\mathbf{r} - \mathbf{r}'\|} \, dS(\mathbf{r}). \tag{6.2.3}$$

Proof. Let D be an open bounded subset of $\mathbb{R}^3$ with smooth boundary. For $u, v \in C^2(\bar{D})$, Green's formula states that

$$\int_D (u\Delta v - v\Delta u) \, d\mathbf{r} = \int_{\partial D} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} \right) dS, \tag{6.2.4}$$

where $\frac{\partial u}{\partial \nu}$ is the outward normal derivative of u on the boundary ∂D of D. We apply (6.2.4) to $D = D_2$, $u = G = G_m^n$, and

$$v(\mathbf{r}) = \frac{1}{4\pi \|\mathbf{r}' - \mathbf{r}\|}. \tag{6.2.5}$$

Since $\Delta u = \Delta v = 0$ on D_2 we obtain

$$0 = \int_{\partial D_2} \left(G\frac{\partial v}{\partial \nu} - v\frac{\partial G}{\partial \nu} \right) dS. \tag{6.2.6}$$

Let $B_r(\mathbf{q})$ denote the open ball centered at $\mathbf{q} \in \mathbb{R}^3$ with radius $r > 0$. We apply (6.2.4) a second time with $D = B_R(\mathbf{0}) - \bar{D}_2 - B_\epsilon(\mathbf{r}')$ with large R and small $\epsilon > 0$. Choose $u = H = H_m^n$ and v as in (6.2.5). Note that $\Delta u = \Delta v = 0$ on D. By a standard argument, taking the limit $\epsilon \to 0$, we obtain

$$H(\mathbf{r}') = \int_{\partial B_R(\mathbf{0})} \left(H \frac{\partial v}{\partial \nu} - v \frac{\partial H}{\partial \nu} \right) dS - \int_{\partial D_2} \left(H \frac{\partial v}{\partial \nu} - v \frac{\partial H}{\partial \nu} \right) dS, \qquad (6.2.7)$$

where, in the second integral, $\frac{\partial}{\partial \nu}$ denotes again the derivative in the direction of the outward normal as in (6.2.6). As $R \to \infty$, the first integral on the right-hand side of (6.2.7) tends to 0 (see [8], page 109), so we obtain

$$H(\mathbf{r}') = -\int_{\partial D_2} \left(H \frac{\partial v}{\partial \nu} - v \frac{\partial H}{\partial \nu} \right) dS. \qquad (6.2.8)$$

Set $E(\zeta) = W^n_{|m|-\frac{1}{2}}(\zeta)$ and $F(\zeta) = W^n_{|m|-\frac{1}{2}}(2K - \zeta)$. We multiply (6.2.6) by $F(\beta_0)$, then multiply (6.2.8) by $E(\beta_0)$ and add these equations. Using the definitions (6.1.2) and (6.2.1) of internal and external flat-ring harmonics and cancelling terms, we find

$$E(\beta_0) H(\mathbf{r}') = \int_{\partial D_2} v \left(E(\beta_0) \frac{\partial H}{\partial \nu} - F(\beta_0) \frac{\partial G}{\partial \nu} \right) dS. \qquad (6.2.9)$$

Since flat-ring coordinates are orthogonal, the normal derivative and the partial derivative with respect to β are related by

$$\frac{\partial}{\partial \nu} = \frac{1}{h_\beta} \frac{\partial}{\partial \beta},$$

where h_α is given in (2.2.5). Let $\mathbf{r} = (x, y, z) \in \partial D_2$ with flat-ring coordinates α, β, ϕ. Then

$$\begin{aligned}
E(\beta_0) & \frac{\partial H}{\partial \nu}(\mathbf{r}) - F(\beta_0) \frac{\partial G}{\partial \nu}(\mathbf{r}) \\
&= E(\beta_0) \frac{\partial((x^2 + y^2)^{-1/4})}{\partial \nu} F(\beta_0) E(\alpha) e^{im\phi} \\
&\qquad + E(\beta_0)(x^2 + y^2)^{-1/4} h_\beta^{-1} F'(\beta_0) E(\alpha) e^{im\phi} \\
&\qquad - F(\beta_0) \frac{\partial((x^2 + y^2)^{-1/4})}{\partial \nu} E(\beta_0) E(\alpha) e^{im\phi} \\
&\qquad - F(\beta_0)(x^2 + y^2)^{-1/4} h_\beta^{-1} E'(\beta_0) E(\alpha) e^{im\phi} \\
&= h_\beta^{-1} (x^2 + y^2)^{-1/4} \left\{ E(\beta_0) F'(\beta_0) - E'(\beta_0) F(\beta_0) \right\} E(\alpha) e^{im\phi}.
\end{aligned}$$

We now use the Wronskian (6.2.2) and obtain

$$E(\beta_0) \frac{\partial H}{\partial \nu}(\mathbf{r}) - F(\beta_0) \frac{\partial G}{\partial \nu}(\mathbf{r}) = \frac{w}{h_\beta(\mathbf{r}) E(\beta_0)} G(\mathbf{r}). \qquad (6.2.10)$$

When we substitute (6.2.10) in (6.2.9) we obtain (6.2.3). $\square$

6.3 The Dirichlet Problem

From the known orthogonality and completeness properties of the Lamé-Wangerin functions, we obtain the following theorem.

Theorem 15. *The system of functions*

$$(8\pi)^{-1/2} W^n_{|m|-\frac{1}{2}}(\alpha, k^2) e^{im\phi}, \quad m \in \mathbb{Z}, n \in \mathbb{N}_0$$

is an orthonormal basis in the Hilbert space

$$H_2 = L^2((iK', 2K + iK') \times (-\pi, \pi)).$$

where $m \in \mathbb{Z}$, $n \in \mathbb{N}_0$.

We use the external peanut harmonics of the second kind to solve the Dirichlet problem for harmonic functions in the region D_2 defined in section 6.1.

Theorem 16. *Let f be a function defined on the boundary ∂D_2 of the region D_2. Suppose that f is represented in flat-ring coordinates as*

$$(x^2 + y^2)^{1/4} f(x, y, z) = g(\alpha', \phi), \quad \alpha' = \alpha + iK', \quad \alpha \in (0, 2K), \quad \phi \in (-\pi, \pi)$$

such that $g \in H_2$. For all $m \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, we define

$$\begin{aligned} c^n_m &:= \frac{1}{8\pi W^n_{|m|-\frac{1}{2}}(\beta_0, k^2)} \int_{-\pi}^{\pi} \int_0^{2K} g(\alpha', \phi) W^n_{|m|-\frac{1}{2}}(\alpha', k^2) e^{-im\phi}\, d\alpha\, d\phi \\ &= \frac{1}{8\pi \left\{ W^n_{|m|-\frac{1}{2}}(\beta_0, k^2) \right\}^2} \int_{\partial D_2} \frac{1}{h_\alpha(\mathbf{r})} f(\mathbf{r}) G^n_{-m}(\mathbf{r})\, dS(\mathbf{r}). \end{aligned}$$

Then the function

$$u(\mathbf{r}) = \sum_{m \in \mathbb{Z}} \sum_{n=0}^{\infty} (c^n_m G^n_m(\mathbf{r})) \tag{6.3.1}$$

is harmonic in D_2 and it assumes the boundary values f on ∂D_2 in the weak sense. The infinite series in (6.3.1) *converges absolutely and uniformly in compact subsets of D_2.*

Proof. Using flat-ring coordinates we can write surface integrals over ∂D_2 as double integrals:

$$\begin{aligned} \int_{\partial D_2} f(\mathbf{r})\, dS(\mathbf{r}) &= \int_0^{2K} \int_{-\pi}^{\pi} h_\alpha h_\phi (x^2 + y^2)^{-1/4} g(\alpha', \phi)\, d\alpha\, d\phi \\ &= \int_0^{2K} \int_{-\pi}^{\pi} h_\alpha (x^2 + y^2)^{1/4} g(\alpha', \phi)\, d\alpha\, d\phi \end{aligned}$$

with the metric coefficients h_α, h_ϕ given in (2.2.5). This shows that the formula given for c^n_m agrees. The rest of the proof is similar to the proof of Theorem 6.3 in [6] . □

6.4 A Fundamental Solution

We obtain the expansion of (6.2.5) in internal and external peanut harmonics by combining Theorems 16 and 14.

Theorem 17. *Let* $\mathbf{r}, \mathbf{r}' \in \mathbb{R}^3$ *with flat-ring coordinates* $\beta, \beta' \in (K - iK', K + iK')$*, respectively. If* $\operatorname{Re}\beta > \operatorname{Re}\beta'$ *then*

$$\frac{1}{\|\mathbf{r} - \mathbf{r}'\|} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \sum_{n=0}^{\infty} \left(G_m^n(\mathbf{r}) H_{-m}^n(\mathbf{r}') \right), \tag{6.4.1}$$

where $m \in \mathbb{Z}$*,* $n \in \mathbb{N}_0$*.*

Proof. We pick $\beta_0 \in (K - iK', K + iK')$ such that $\operatorname{Re}\beta' < \operatorname{Re}\beta_0 < \operatorname{Re}\beta$, and consider the domain D_2 defined in section 6.1. The function $f(\mathbf{q}) := \frac{1}{\|\mathbf{q}-\mathbf{r}'\|}$ is harmonic on an open set containing $\bar{D}_2$. Therefore, by Theorem 16, we have

$$f(\mathbf{r}) = \sum_{m \in \mathbb{Z}} \sum_{n=0}^{\infty} \left(c_m^n G_m^n(\mathbf{r}) \right), \tag{6.4.2}$$

where c_m^n can be evaluated by Theorem 14:

$$c_n^m = \frac{1}{2} \mathrm{Hc}_{-m}^n(\mathbf{r}').$$

We used $h_{\beta'} = h_\alpha$. Thus we obtain (6.4.1). $\square$

Bibliography

[1] H. Volkmer, Unpublished manuscript, 2014.

[2] H. Cohl, Unpublished manuscript, 2011.

[3] O. Kellog, *Foundations of Potential Theory*, Dover reprint, Dover Publications, New York, 1954.

[4] W. Miller, Jr. *Symmetry and separation of variables.* Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1977. With a foreword by Richard Askey, Encyclopedia of Mathematics and its Applications, Vol. 4.

[5] N. N. Lebedev. *Special functions and their applications.* Dover Publications Inc., New York, 1972. Revised edition, translated from the Russian and edited by Richard A. Silverman, Unabridged and corrected republication.

[6] H. Cohl and H. Volkmer. Separation of variables in an asymmetric cyclidic coordinate system, Math. Phys, (54), 063513, 2013.

[7] H. Cohl and H. Volkmer. Expansions for a fundamental solution of Laplace's equation on $\mathbb{R}^3$ in 5-cyclidic harmonics. Analysis and Applications, Vol. 12, No.6, 613-633, 2014.

[8] R. McOwen. *Partial Differential Equations: Methods and Applications.* Prentice-Hall, Inc, 1995.

[9] M. Bôcher. *Ueber die Reihenentwickelungen der Potentialtheorie.* B. G. Teubner, Leipzig, 1894.

[10] P. M. Morse and H. Feshbach. *Methods of theoretical physics. 2 volumes.* McGraw-Hill Book. Co., Inc., New York, 1953.

[11] J. Blimke, J. Myklebust, H. Volkmer, and S. Merrill. Four-shell ellipsoidal model employing multipole expansion in ellipsoidal coordinates. Medical & Biological Engineering & Computing, 46(9):859–869, Sep 2008.

[12] E. Heine *Handbuch der Kugelfunctionen, Theorie und Anwendungen.* Druck und Verlag von G. Reimer, Berlin, 1881.

[13] A. Erdélyi, Higher transcendental functions, Vol. III, McGraw-Hill, New York, 1955.

[14] E. G. C. Poole, Dirichlet's principle for a flat ring, Proc. London Mat. Soc. 29 (2) (1929), 342-354.

[15] E. G. C. Poole, Dirichlet's principle for a flat ring (second paper), Proc. London Mat. Soc. 30 (2) (1950), 174-186.

[16] R. Miller and A. Michel, Ordinary differential equations, Academic Press, Orlando, FL, 1982.

[17] W.H. McCrea and R. A. Newing, Boundary conditions for the wave equation, London Math. Society s2-37 (1934), 520-534.

[18] K. Jörgens and K. Rellich, Eigenwerttheorie gewöhnlicher Differentialgleichungen, Springer-Verlag, Berlin-Heidelberg-New York 1976.

[19] W. Walter, Introduction to ordinary differential equations, Springer-Verlag.

[20] A. Zettl, Sturm-Liouville Theory, Mathematical Surveys and Monographs, Volume 121, American Mathematical Society, Providence 2005.

[21] W. Magnus and S. Winkler, Hill's equation, Dover Publications, Inc., New York, 1979.

[22] E. T. Whittaker and G. N. Waston, A course of modern analysis, Cambridge University Press, Cambridge, 1996.

Appendix

General Descriptions of the Functions sn$u$, cn$u$, dn$u$

Let $0 < k^2 < 1$,

$$K(k) := \int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}\,\sqrt{1-k^2x^2}},$$
$$K'(k) := K(k').$$

(I) The function sn$u$ is a doubly-periodic function of u with periods $4K$, $2iK'$. It is analytic except at the points congruent to iK' or to $2K + iK'$ (mod. $4K$, $2iK'$); these points are simple poles, the residues at the first set all being k^{-1} and the residues at the second set all being $-k^{-1}$; and the function has a simple zero at all points congruent to 0 (mod. $2K$, $2iK'$).

(II) The function cn$u$ is a doubly-periodic function of u with periods $4K$ and $2K+2iK'$. It is analytic except at points congruent to iK' or to $2K + iK'$ (mod. $4K$, $2K + 2iK'$; these points are simple poles, the residues at the the first sent being $-ik^{-1}$, and the residues at the second set being ik^{-1}; and the function has a simple zero at all points congruent to K (mod. $2K$, $2iK'$).

(III) The function dn$u$ is a doubly-periodic function of u with periods $2K$ and $4iK'$. It is analytic except at points congruent to iK' or to $3iK'$ (mod. $2K$, $4iK'$); these points are simple poles, the residues at the first set being $-i$, and the residues at the second set being i; and the function has a simple zero at all points congruent to $K + iK'$ (mod. $2K$, $2iK'$).

It may be observed that sn$u$, cn$u$, and dn$u$ are the only functions satisfying the above descriptions, respectively, see [22, page 504].

Lamé-Wangerin Functions

The eigenvalue problem

We consider the differential equation

$$-y'' + q(x)y = \lambda y, \quad 0 < x \le b < \infty. \tag{6.4.3}$$

We assume that $q : (0, b] \to \mathbb{R}$ is a continuous function. Additionally, we assume that $q(x)$ has the Laurent series representation

$$q(x) = \frac{1}{x^2} \sum_{k=0}^{\infty} q_k x^k, \quad 0 < x < \epsilon, \tag{6.4.4}$$

for some $\epsilon > 0$. The differential equation (6.4.3) has a regular singularity at $x = 0$ with indicial equation

$$r(r-1) - q_0 = 0.$$

The solutions (called exponents) of this equation are

$$r_1 = \frac{1}{2} - \sqrt{\frac{1}{4} + q_0}, \quad r_2 = \frac{1}{2} + \sqrt{\frac{1}{4} + q_0}.$$

If $q_0 = -\frac{1}{4}$ we have a double root $r_1 = r_2 = \frac{1}{2}$. If $q_0 > -\frac{1}{4}$, there are two distinct real roots $r_1 < \frac{1}{2} < r_2$. If $q_0 < -\frac{1}{4}$, the roots r_1, r_2 are conjugate complex.

For every $\lambda \in \mathbb{C}$, equation (6.4.3) has a fundamental system $y_1(x, \lambda), y_2(x, \lambda)$ with the property

$$\begin{aligned} y_1(x,\lambda) &= x^{r_1} \sum_{k=0}^{\infty} c_k x^k + c y_2(x,\lambda) \ln x, \quad 0 < x < \epsilon, \\ y_2(x,\lambda) &= x^{r_2} \sum_{k=0}^{\infty} d_k x^k, \quad 0 < x < \epsilon. \end{aligned}$$

If $r_2 - r_1 \neq 0, 1, 2, \ldots$ then $c_0 = d_0 = 1$, $c = 0$. If $r_1 = r_2$ then $c_0 = 0$, $d_0 = 1$, $c = -1$. If $r_2 - r_1 = m_0$ a positive integer, then $c_0 = d_0 = 1$, $c_{m_0} = 0$. This is a well-known result; see [18, Satz 1, page 147]. We have

$$d_0 = 1, \quad d_1 = \frac{q_1}{2r_2}, \quad d_2 = \frac{2r_2 q_2 - 2\lambda r_2 + q_1^2}{4r_2(2r_2+1)}.$$

In the following we assume that

$$q_0 \ge -\frac{1}{4}.$$

Let $0 < \beta \leq \pi$. We call $\lambda \in \mathbb{C}$ an eigenvalue if

$$\cos\beta\, y_2(b,\lambda) = \sin\beta\, y_2'(b,\lambda), \quad {}' = \frac{d}{dx}. \tag{6.4.5}$$

We call $y_2(x,\lambda)$ a corresponding eigenfunction.

Main results

In this section we argue essentially the same way as in the theory of regular Sturm-Liouville problems.

Lemma 1. *For every $\lambda, \mu \in \mathbb{C}$ we have*

$$\lim_{x\to 0^+} y_2(x,\lambda)y_2'(x,\mu) = \begin{cases} 0 & \text{if } q_0 > -\frac{1}{4}, \\ \frac{1}{2} & \text{if } q_0 = -\frac{1}{4}. \end{cases}$$

Proof. If $q_0 > -\frac{1}{4}$ then $r_2 > \frac{1}{2}$ and if $q_0 = -\frac{1}{4}$ then $r_2 = \frac{1}{2}$. □

Lemma 2. *If λ_1, λ_2 are two distinct real eigenvalues, then*

$$\int_0^b y_2(x,\lambda_1)y_2(x,\lambda_2)\,dx = 0.$$

Proof. Let $u_j(x) = y_2(x,\lambda_j)$. For $0 < \delta < b$ we obtain

$$\begin{aligned}(\lambda_1 - \lambda_2)\int_\delta^b u_1(x)u_2(x)\,dx &= [u_1(x)u_2'(x) - u_1'(x)u_2(x)]_\delta^b \\ &= -u_1(\delta)u_2'(\delta) + u_1'(\delta)u_2(\delta).\end{aligned}$$

By Lemma 1, as $\delta \to 0^+$,

$$(\lambda_1 - \lambda_2)\int_0^b u_1(x)u_2(x)\,dx = 0.$$

Since $\lambda_1 \neq \lambda_2$ the desired statement follows. □

Lemma 3. *All eigenvalues are real.*

Proof. Let λ be a complex eigenvalue. Setting $u_1 = y_2(x,\lambda)$ and $u_2 = y_2(x,\bar{\lambda})$ we obtain arguing as in the proof of Lemma 2,

$$2i\,\mathrm{Im}\,\lambda \int_0^b |u_1(x)|^2\,dx = 0.$$

Therefore, $\mathrm{Im}\,\lambda = 0$. □

We introduce the Prüfer angle $\theta(x,\lambda)$ and Prüfer radius $r(x,\lambda)>0$ for $0<x\le b$ by setting

$$y_2(x,\lambda)=r(x,\lambda)\sin\theta(r,\lambda),\quad y_2'(x,\lambda)=r(x,\lambda)\cos\theta(r,\lambda).$$

The Prüfer angle is defined to be between 0 and $\pi/2$ for small positive x and by continuity otherwise. For small positive x we have

$$\theta(x,\lambda)=\arctan\frac{y_2(x.\lambda)}{y_2'(x,\lambda)}.$$

This shows that $\theta(x,\lambda)$ is an analytic function of x in a neighborhood of 0. We calculate

$$\theta(x)=a_1x+a_2x^2+a_3x^3+\dots \tag{6.4.6}$$

with

$$a_1=\frac{1}{r_2},\quad a_2=-\frac{q_1}{2r_2^3},$$
$$a_3=\frac{12\lambda r_2^3-12r_2^3q_2-8r_2^3+9q_1^2r_2-4r_2^2+3q_1^2}{12r_2^5(2r_2+1)}.$$

The Prüfer angle satisfies the first order differential equation

$$\theta'=\cos^2\theta+(\lambda-q(x))\sin^2\theta. \tag{6.4.7}$$

The following lemma collects properties of $\theta(x,\lambda)$. These properties are analogous to those of the Prüfer angle for regular Sturm-Liouville problems; see [19].

Lemma 4. *(a) When $\theta(x_0,\lambda)=n\pi$ for some integer n and $0\le x_0\le b$, then $\theta(x,\lambda)<n\pi$ for $0<x<x_0$ and $\theta(x,\lambda)>n\pi$ for $x_0<x\le b$.*
(b) The function $\lambda\to\theta(x,\lambda)$ is continuous and strictly increasing for every $0<x\le b$.
(c) $\lim_{\lambda\to-\infty}\theta(b,\lambda)=0$.
(d) $\lim_{\lambda\to\infty}\theta(b,\lambda)=\infty$.

Proof. (a) If $\theta(x_0,\lambda)=n\pi$ then (6.4.7) implies $\theta'(x_0,\lambda)=1$.
(b) Let $\lambda_1<\lambda_2$. The expansion (6.4.6) shows that $\frac{d^k}{dx^k}\theta(0,\lambda_1)=\frac{d^k}{dx^k}\theta(0,\lambda_2)$ for $k=0,1,2$ but $\frac{d^3}{dx^3}\theta(0,\lambda_1)<\frac{d^3}{dx^3}\theta(0,\lambda_2)$. Therefore, we can choose $0<\delta<x$ such that $\theta(\delta,\lambda_1)<\theta(\delta,\lambda_2)$. Then $\theta(x,\lambda_1)<\theta(x,\lambda_2)$ follows from a standard differential inequality; see [16, Theorem 2.1, page 144].
(c) The expansion (6.4.6) shows that there is $0<\delta<b$ such that $0<\theta(\delta,0)<\frac{\pi}{2}$. Let $\Theta(x,\lambda)$ be the solution of (6.4.7) with initial condition $\Theta(\delta,\lambda)=\frac{\pi}{2}$. It follows from (b) that $\theta(\delta,\lambda)<\frac{\pi}{2}$ for $\lambda\le 0$. Therefore, $0<\theta(b,\lambda)<\Theta(b,\lambda)$ for $\lambda\le 0$. It is known from regular Sturm-Liouville theory that $\lim_{\lambda\to-\infty}\Theta(b,\lambda)=0$. This proves (c).
(d) We choose $0<\delta<b$. Let $\Theta(x,\lambda)$ be the solution of (6.4.7) determined by $\Theta(\delta,\lambda)=0$. Then $\Theta(b,\lambda)<\theta(b,\lambda)$ for all real λ. It is known from regular Sturm-Liouville theory that $\lim_{\lambda\to\infty}\Theta(b,\lambda)=\infty$. This proves (d). □

Theorem 18. *The eigenvalues (solutions of* (6.4.5)*) form an increasing sequence*

$$\lambda_0 < \lambda_1 < \lambda_2 < \dots$$

converging to infinity. The corresponding eigenfunctions $y_2(x, \lambda_n)$ *have exactly* n *zeros in the open interval* $(0, b)$.

Proof. By Lemma 3, all eigenvalues are real. A real number λ is an eigenvalue if and only if there exists $n = 0, 1, 2, \dots$ such that

$$\theta(b, \lambda) = \beta + n\pi. \tag{6.4.8}$$

By Lemma 4, for every $n = 0, 1, 2, \dots$, equation (6.4.8) has exactly one solution $\lambda = \lambda_n$. By Lemma 4(a), $y_2(x, \lambda_n)$ has exactly n zeros in $(0, b)$. □

Let $C[0, b]$ be the vector space of continuous real-valued functions defined on $[0, b]$ endowed with the inner product

$$\langle f, g\rangle = \int_0^b f(x)g(x)\, dx.$$

This is a pre-Hilbert space. We could also work in the Hilbert space $L^2(0, b)$ but we want to avoid usage of the Lebesgue integral in this section. An orthogonal sequence $\{f_n\}$, $f_n \neq 0$, in a pre-Hilbert space H is called complete if the closed linear span of the sequence is H. The sequence is complete if and only if every function $f \in H$ can be written as a generalized Fourier series

$$f = \sum_n \frac{\langle f, f_n\rangle}{\langle f_n, f_n\rangle} f_n$$

with convergence in H.

By Lemma 2, the sequence $\{y_2(x, \lambda_n)\}_{n=0}^{\infty}$ is an orthogonal sequence in $C[0, b]$. We are going to show that this sequence is complete (this result is missing in [17].).

For $\lambda \in \mathbb{C}$ let $y_3(x, \lambda)$ be the solution of (6.4.3) determined by initial conditions

$$y_3(b, \lambda) = \sin\beta, \quad y_3'(b, \lambda) = \cos\beta.$$

λ is an eigenvalue if and only if y_2 and y_3 are linearly dependent. Choose $\lambda^* \in \mathbb{R}$ such that λ^* is not an eigenvalue, and define the Green's function

$$G(x, t) = W^{-1}\begin{cases} y_2(x)y_3(t), & 0 < x < t \le b, \\ y_2(t)y_3(x), & 0 < t \le x \le b, \end{cases}$$

where $y_j(x) = y_j(x, \lambda^*)$ for $j = 2, 3$, and W is the Wronskian

$$W = y_3y_2' - y_3'y_2.$$

W is a nonzero constant.

Lemma 5. *If we set $G(x,t) = 0$ for $x = 0$ or $t = 0$, then $G : [0,b] \times [0,b] \to \mathbb{R}$ is continuous.*

Proof. We use the estimates

$$|y_2(x)| \le B_2 x^{r_2}, \quad |y_3(x)| \le B_3 x^{r_1} |\ln x|, \quad \text{for } 0 < x \le b,$$

where B_2, B_3 are constants. If $0 < t \le x \le b$, we have

$$|y_2(t) y_3(x)| \le B_2 B_3 t^{r_2} x^{r_1} \le B_2 B_3 x^{r_1+r_2} |\ln x| = B_2 B_3 x |\ln x|.$$

This shows that G is continuous on the triangle $0 \le x \le t \le b$. Since $G(x,t) = G(t,x)$, G is continuous on $[0,b]^2$. $\square$

We consider the integral operator $T : C[0,b] \to C[0,b]$ defined by

$$Tf(x) = \int_0^b G(x,t) f(t)\, dt.$$

Since G is continuous and $G(x,t) = G(t,x)$, we have the following expansion theorem; see [19].

Theorem 19. *The operator T has a sequence of nonzero eigenvalues $\{\mu_n\}$ with corresponding orthogonal eigenfunctions f_n such that the closed linear span of $\{f_n\}$ contains the range of T.*

Lemma 6. *λ is an eigenvalue in the sense of (6.4.5) if and only if $(\lambda - \lambda^*)^{-1}$ is an eigenvalue of T. The corresponding eigenfunctions are the same.*

Proof. Let $f \in C[0,b]$ and set $y = Tf$. Then, for $0 < x \le b$,

$$y(x) = W^{-1} \left(\int_0^x y_2(t) f(t)\, dt \right) y_3(x) + W^{-1} \left(\int_x^b y_3(t) f(t)\, dt \right) y_2(x).$$

Differentiating we obtain

$$y'(x) = W^{-1} \left(\int_0^x y_2(t) f(t)\, dt \right) y_3'(x) + W^{-1} \left(\int_x^b y_3(t) f(t)\, dt \right) y_2'(x).$$

Differentiating once more we find

$$y''(x) = -f(x) + (q(x) - \lambda^*) y(x).$$

If $Tf = (\lambda - \lambda^*)^{-1} f$ with $f \ne 0$ then setting $y = Tf \ne 0$ we obtain $-y'' + q(x)y = \lambda y$. We have

$$y(b) = W^{-1} \left(\int_0^b y_2(t) f(t)\, dt \right) y_3(b), \quad y'(b) = W^{-1} \left(\int_0^b y_2(t) f(t)\, dt \right) y_3'(b).$$

Therefore, y satisfies the boundary condition $\cos\beta y(b) = \sin\beta y'(b)$. If $r_1 < 0$ then y must be a constant multiple of $y_2(x, \lambda_n)$ for some n because $y = Tf$ is continuous on $[0, b]$. Therefore, $\lambda = \lambda_n$ for some $n = 0, 1, 2 \dots$ If $r_1 \ge 0$ then $y(x) = O(x) + O(x^{r_2})$ as $x \to 0^+$ so we reach the same conclusion.

Conversely, we wish to show that $Ty_2(x, \lambda_n) = (\lambda_n - \lambda^*)^{-1} y_2(x, \lambda_n)$. This follows from the equations

$$\int_0^x y_2(t) y_2(t, \lambda_n)\, dt = (\lambda_n - \lambda^*)^{-1} (y_2(x, \lambda_n) y_2'(x) - y_2'(x, \lambda_n) y_2(x)),$$
$$\int_x^b y_3(t) y_2(t, \lambda_n)\, dt = (\lambda_n - \lambda^*)^{-1} (-y_2(x, \lambda_n) y_3'(x) + y_2'(x, \lambda_n) y_3(x)).$$

□

Theorem 20. *The eigenfunctions* $\{y_2(x, \lambda_n)\}_{n=0}^{\infty}$ *are complete in* $C[0, b]$.

Proof. It follows from Lemma 6 and Theorem 19 that the closed linear span of the eigenfunctions $y(x, \lambda_n)$ contains the range of T. Therefore, it is enough to show that the range of T is dense in $C[0, b]$. Let $g \in C^2[0, b]$ with compact support in $(0, b)$. Then $f(x) := -g''(x) + (q(x) - \lambda^*) g(x)$ is continuous on $[0, b]$ with compact support in $(0, b)$. The function $y = Tf$ satisfies $f(x) = -y''(x) + (q(x) - \lambda^*) y(x)$, so $v(x) := g(x) - y(x)$ satisfies $-v''(x) + (q(x) - \lambda^*) v(x) = 0$. Now $\cos\beta\, v(v) = \sin\beta\, v'(b)$, and $y(x)$ is a constant multiple of $y_2(x)$ for x close to 0. Therefore, $g = y$ and g is in the range of T. The set of functions $g \in C^2[0, b]$ with compact support in $(0, b)$ is dense in $C[0, b]$. The proof is complete. □

Connectios to Weyl's theory of singular Sturm-Liouville problems

We see that $y_2 \in L^2(0, \frac{1}{2}\epsilon)$ for all $q_0 \in \mathbb{R}$, and $y_1 \in L^2(0, \frac{1}{2}\epsilon)$ if and only if $q_0 < \frac{3}{4}$. From this observation we obtain the following lemma.

Lemma 7. *Equation* (6.4.3) *is in the limit-point case at* $x = 0$ *if and only if* $q_0 \ge \frac{3}{4}$.

Lemma 7 is a special case of [18, Satz 1, page 152].

The point b is a regular end point. At this end point we pose the boundary condition

$$\cos\beta\, y(b) = \sin\beta\, y'(b). \tag{6.4.9}$$

If $q_0 \ge \frac{3}{4}$ we do not need a boundary condition at $x = 0$. If $-\frac{1}{4} \le q_0 < \frac{3}{4}$, we add a boundary condition that forces eigenfunctions to be constant multiples of y_2. Let $H = L^2(0, b)$. If $q_0 \ge \frac{3}{4}$, we introduce the linear operator $A : H \supset D(A) \to H$ by $Ay = -y'' + qy$ with domain

$$D(A) = \{y : y, y' \in AC[\delta, b] \text{ for all } 0 < \delta < b, y, -y'' + qy \in H, y \text{ satisfies (6.4.9)}\}.$$

If $-\frac{1}{4} \leq q_0 < \frac{3}{4}$, we add the boundary condition

$$\lim_{x \to 0^+} (y_2(x)y'(x) - y_2'(x)y(x)) = 0.$$

It is known that A is a self-adjoint operator. For the operator $T : H \to H$ defined in Section 2 we have

$$T = (A - \lambda^*)^{-1}.$$

Since T is compact, A is a self-adjoint operator with compact resolvent. We also see that A is bounded below. The eigenvalues of A are exactly the numbers λ_n, $n = 0, 1, 2, \ldots$

Examples

Example 3: The Lamé-Wangerin functions are eigenfunctions of the differential equation

$$-y'' + \frac{\nu(\nu+1)}{\operatorname{sn}^2(x,k)} y = \lambda y.$$

Then $q_0 = \nu(\nu+1)$, $q_1 = 0$, $q_2 = \nu(\nu+1)\frac{1}{3}(k^2+1)$. The boundary condition is $y(K) = 0$ or $y'(K) = 0$.

Lijuan Bi

Education

Ph.D, Mathematics
University of Wisconsin-Milwaukee Milwaukee, WI, USA
August 2018
Advisor: Professor Hans Volkmer
Dissertation: Interior and Exterior Flat-Ring Harmonics

M.S, Mathematics, Summa Cum Laude
University of Texas-Pan American Edinburg, TX, USA
August 2010
Advisor: Professor Arunava Mukherjea
Thesis : Identification of Parameters and the Distribution of the Maximum and the Minimum

B.Sc, Mathematics
Liaoning University Shenyang, Liaoning, China August 2008

Teaching Positions

Graduate Student Teaching
University of Wisconsin-Milwaukee Milwaukee, WI, USA 2010-May 2018

Graduate Student Teaching
University of Texas-Pan American Edinburg, TX, USA 2008-2010

Research Interests

Ordinary Differential Equations; Partial Differential Equations; Special Functions

Publications

Bi, L.; Mukherjea, A. Poisson distributions: identification of parameters from the distribution of the maximum and a conjecture on the partial sums of the power series for exp(x). Statistics & Probability Letters. 81 (2011), no. 5, 611-613.

Bi, L.; Mukherjea, A. Identification of parameters and the distribution of the minimum of the tri-variate normal. Statistics & Probability Letters. 80 (2010), no. 23-24, 1819-826.

Interior and Exterior Flat-Ring Harmonics (in preparation)

Selected Honors and Distinctions

Graduate Student Travel Award, 2017
$1,000 to attend mathematics joint meetings in January 2018 at San Diego.

Research Excellence Award, 2016
$1,000 per year for graduate student who exhibits outstanding research potential.

Ernst Schwandt Teaching Award, 2016
$250 award for Graduate Teaching Assistants who demonstrate outstanding teaching performance.
Only 4 Graduate Teaching Assistants out of 90 won this award in 2016.

Chancellor's Fellowship Award, 2010-2015
$4,000 per year for years 2010-2012, $3,000 for 2013, $2,000 for 2014, and $1,000 for 2015 for
graduate students with exceptional academic records with high promise of future success.